

On the Existence of Envy-Free Allocations Beyond Additive Valuations

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Abstract

We study the problem of fairly allocating m indivisible items among n agents. Envy-free allocations, in which each agent prefers her bundle to the bundle of every other agent, need not exist in the worst case. However, when agents have additive preferences and the value $v_{i,j}$ of agent i for item j is drawn independently from a distribution D_i , envy-free allocations exist with high probability when $m \in \Omega(n \log n / \log \log n)$.

In this paper, we study the existence of envy-free allocations under stochastic valuations far beyond the additive setting. We introduce a new stochastic model in which each agent’s valuation is sampled by first fixing a worst-case function, and then drawing a uniformly random renaming of the items, independently for each agent. This strictly generalizes known settings; for example, $v_{i,j} \sim D_i$ may be seen as picking a random (instead of a worst-case) additive function before renaming. We prove that random renaming is sufficient to ensure that envy-free allocations exist with high probability in very general settings. When valuations are non-negative and “order-consistent,” a valuation class that generalizes additive, budget-additive, unit-demand, and single-minded agents, SD-envy-free allocations (a stronger notion of fairness than envy-freeness) exist for $m \in \omega(n^2)$ when n divides m , and SD-EFX allocations exist for all $m \in \omega(n^2)$. The dependence on n is tight, that is, for $m \in O(n^2)$ envy-free allocations don’t exist with constant probability. For the case of arbitrary valuations (allowing non-monotone, negative, or mixed-manna valuations) and $n = 2$ agents, we prove envy-free allocations exist with probability $1 - \Theta(1/m)$ (and this is tight).

1 Introduction

We consider the fundamental problem of fairly allocating a set \mathcal{M} of m indivisible items among a set \mathcal{N} of n agents. Each agent i has a valuation function $v_i : 2^{\mathcal{M}} \mapsto \mathbb{R}$, which maps each subset of items $S \subseteq \mathcal{M}$ to a value for S . In this domain, the gold standard of fairness is, arguably, *envy-freeness*. An allocation $\mathcal{A} = (A_1, A_2, \dots, A_n)$ is envy-free if each agent prefers her own bundle to the bundle of every other agent, that is, $v_i(A_i) \geq v_i(A_j)$ for all $i, j \in \mathcal{N}$. It is easy to see that envy-free allocations do not exist in the worst case: consider a single item and two agents valuing it positively.

Motivated to circumvent this simple non-existence result, a line of research in fair division studies the existence of envy-free allocations under stochastic valuations. To date, this work has focused on additive valuation functions.¹ Dickerson et al. [DGK⁺14] show that when agents have additive and non-negative valuation functions, and all item values are drawn independently from a distribution \mathcal{D} , allocations that simultaneously satisfy envy-freeness and Pareto efficiency exist with high probability for $m \in \Omega(n \log n)$. On the other hand, envy-free allocations do not exist with constant probability for $m \in n + o(n)$. In the same

¹A valuation function $v_i : 2^{\mathcal{M}} \mapsto \mathbb{R}$ is additive if $v_i(S \cup T) = v_i(S) + v_i(T)$ for all $S, T \subseteq \mathcal{M}$, $S \cap T = \emptyset$. An additive function v_i can be succinctly represented with a value $v_{i,j}$ for each item $j \in \mathcal{M}$, such that $v_i(S) = \sum_{j \in S} v_{i,j}$ for all $S \subseteq \mathcal{M}$ and $i \in \mathcal{N}$.

setting, Manurangsi and Suksompong [MS20] show that an envy-free allocation exists with high probability as long as $m \geq 2n$ and n divides m . When m is not “almost divisible” by n ,² an envy-free allocation is unlikely to exist for $m \in O(n \log n / \log \log n)$. Manurangsi and Suksompong [MS21] close this gap by proving that envy-free allocations exist with high probability when $m \in \Omega(n \log n / \log \log n)$. More recently, Bai and Gözl [BG22] extend these bounds ($m \in \Omega(n \log n / \log \log n)$ for envy-freeness and $\Omega(n \log n)$ for envy-freeness plus Pareto efficiency) to the additive non-i.i.d. case where item values drawn independently from (agent specific) distributions \mathcal{D}_i .

We investigate the existence of envy-free allocations under stochastic valuations beyond the additive case.

1.1 Our Contribution

Before asking if envy-free allocations exist for valuations beyond additive, it is necessary to specify a stochastic model for generating such valuations. “Beyond additive” models are common in mechanism design in the context of maximizing expected value or revenue. For example, in the literature on prophet inequalities (see [Luc17] for a recent survey) valuation functions are typically drawn from distributions over families of functions (e.g. submodular or XOS functions). In the literature on auctions, the predominant “beyond additive” model is the “ \mathcal{C} over independent items” model for a condition \mathcal{C} (e.g. “subadditive over independent items”), introduced by [RW18], where, informally, an agent’s valuation function is parameterized by a vector of types, drawn from a product distribution.³ In both these models, however, it is easy to reconstruct the simple “two agents, one item” counterexample, e.g. by picking a valuation function/distribution over valuation functions such that item 1 has more value than all other items combined. And, importantly, this counterexample does not go away as the number of items grows. To bypass such trivial lower bounds, we need a stochastic model that is *neutral* with respect to items, that is, there is no a priori “discrimination” between items.

Our model. Our first contribution is to introduce such a neutral model for stochastic valuations. First, fix a worst-case valuation function for each agent. Then, rename the items uniformly at random and independently across agents. Slightly more formally, fix a worst-case valuation function $v_i : 2^{\mathcal{M}} \rightarrow \mathbb{R}$ for agent i , and sample a uniformly random permutation $\pi_i : \mathcal{M} \rightarrow \mathcal{M}$. The valuation $v_i^{\pi_i}(S)$ for a subset of items S after renaming is equal to $v_i(\pi_i^{-1}(S))$. To see that this model generalizes the standard stochastic additive setting where $v_{i,j} \sim D_i$, observe that the distribution over values is invariant with respect to taking a random permutation of the items (renaming). Accordingly, our positive results imply positive results in the old setting (among others).

Initially, it might appear that our model should still allow trivial lower bounds, since renaming does not tell us anything about how *values* for bundles concentrate, and (to the best of our knowledge) the concentration of values was necessary for all previous “envy-free with high probability” results. For example, consider an identical additive function ($v_{i,j} = 1$ for all $i \in \mathcal{N}, j \in \mathcal{M}$), where random renaming has no power. For this function, a necessary condition for envy-free allocations to exist is that m is divisible by n . For an arbitrary worst-case function, one may naturally expect additional conditions (beyond divisibility) for envy-free allocations to exist with high probability. Surprisingly, this is not the case: divisibility is *sufficient* for strong positive results in this model.

EF allocations under order-consistent valuations. We first consider a structured valuation class that strictly generalizes additive valuations in Section 3. Given an order over the items π , we say that a subset of items A stochastically dominates a subset of items B , denoted as $A \succeq_{\pi}^{\text{sd}} B$, when the best item in A (according to π) is weakly better than the best item in B , the second best item in A is weakly better than the second best item in B , and so on. We say that a valuation function v is *order-consistent* with respect to π if, for all bundles $A, B \subseteq \mathcal{M}$, $A \succeq_{\pi}^{\text{sd}} B$ implies that $v(A) \geq v(B)$. By picking π to be the items sorted in order of decreasing value, it is clear that additive valuations are order-consistent. Similarly, budget

²Formally, when the remainder of the division is not between n^{ϵ} and $n - n^{\epsilon}$ for some constant $\epsilon \in (0, 1)$.

³See [RW18, RS17, CZ17, CM16] for more details about this model.

additive,⁴ single-minded,⁵ and unit-demand⁶ valuations are also order-consistent. This valuation class was also considered by Bouveret et al. [BEL⁺10], who study various algorithmic and complexity questions (in the worst-case model).

We prove that, given n arbitrary order-consistent valuation functions over m items, where m is divisible by n , the probability that an envy-free allocation exists after random renaming is at least $1 - O\left(\frac{n^2}{m} + \frac{n \log m}{m^{\frac{n-1}{n}}}\right)$ (Theorem 1). In fact, we prove the existence of a stronger notion of fairness, SD-envy-freeness: an allocation \mathcal{A} is SD-envy-free if, for all agents $i, j \in \mathcal{N}$, $A_i \succeq_{\pi_i}^{\text{sd}} A_j$. As a corollary, we get that for $m \in \omega(n^2)$, SD-envy-free allocations exist with high probability. By “high probability” we mean that for all n and $\delta > 0$, there exists a $m_0 = m_0(n, \delta)$ such that, for all $m > m_0$, the probability is at least $1 - \delta$.

Our proof of Theorem 1 is constructive; we show that a simple Round-Robin process (agents take turns picking the best, according to their order, available item) produces such an allocation. Specifically, for arbitrary agents $i, j \in \mathcal{N}$, we upper bound the probability that i does not sd-prefer their bundle over j ’s, i.e. $A_i \not\succeq_{\pi_i}^{\text{sd}} A_j$. For this to happen, there must be some $1 \leq k \leq m/n$ such that i prefers j ’s k -th best item (according to i) over i ’s k -th best item. At a high level, we’d like to compute this probability, and then take a union bound over k (and then another union bound over all pairs of agents). Notice that for $k = 1$ this probability is already $\Theta(n/m)$, so, in order to afford all the union bounds, it better be the case that the true probabilities of the bad events are much smaller than n/m , and that our analysis is relatively tight. For all items $k < m/n$, i.e. all items except the last one picked by i , we can directly upper bound the probability that i prefers j ’s k -th best item. The analysis leverages the insight that, from i ’s perspective, and over the random draws of π_{-i} , items picked by other agents look like (uniformly) random selections from the pool of remaining items; therefore, the distribution of other agents’ bundles is identical. For i to prefer j ’s k -th best item over their own, j must have selected k items all better than i ’s k -th pick, which can only occur if these items have all been picked in rounds 1 through k ; this event is unlikely. The precise bound is $\Theta(1/\binom{m/n}{k})$ (see Lemma 4), and requires carefully accounting for the items remaining at each step of Round-Robin, coupled with careful applications of known facts about the gamma function (e.g. Gautchi’s inequality [Gau59]). When $k = m/n$, previous arguments fail to yield a sufficiently small probability. Instead, define L as the set of (roughly) $3n \log m$ worst items of agent i . We show that neutrality implies that, with high probability, by the time i picks their m/n -th and worst item, all items in L have been picked by others. When this happens, i must like their worst item more than the worst (from i ’s perspective) item of every agent who picked an item in L . Moreover, conditioned on all items in L being picked by the time of i ’s last pick, it is very likely that every other agent received at least one of these items in L .

Since $|A_i| > |A_j|$ immediately implies that $A_j \not\succeq_{\pi_j} A_i$, m being divisible by n is a necessary condition for envy-freeness. Moreover, standard birthday paradox arguments imply (even for unit-demand valuations) that envy-free allocations may not exist with constant probability for $m \in O(n^2)$. This shows Theorem 1 is tight both in terms of the divisibility assumption and the bound on m . When n does not divide m , we guarantee a weaker notion of fairness called SD-EFX⁷⁸, again with probability at least $1 - O\left(\frac{n^2}{m} + \frac{n \log m}{m^{\frac{n-1}{n}}}\right)$.

Though the relation between m and n is asymptotically tight in Theorem 1, we show that it is possible to get better probability bounds for the important case when n is small, i.e., when n is a constant. Taking, for example, the case of $n = 2$, Round-Robin will not find an SD-EF allocation with probability $\Omega(1/\sqrt{m})$, while a careful analysis would say that such allocations don’t exist with probability $\Omega(1/m)$. This difference is caused by the fact that if n is small, the last item in Round-Robin is given to the “wrong” person. This motivates a new algorithm to close this gap. Our algorithm, “Give-Away Round-Robin,” initially has every agent give each other agent their least desired remaining item and then proceeds with the standard Round-Robin algorithm. The probability of Round-Robin failing is greatest at the beginning and end of the process:

⁴A valuation is budget additive with budget B if each item j has value v_j and the value of $S \subseteq \mathcal{M}$ is $v(S) = \min\{B, \sum_{j \in S} v_j\}$.

⁵A valuation is single-minded if there is a subset S^* , such that $v(S) = v(S^*) > 0$ for all $S \supseteq S^*$, and $v(S) = 0$ otherwise.

⁶A valuation is unit-demand if there is a value v_j for each item j , and the value for a subset of items S is equal to $\max_{j \in S} v_j$.

⁷In an EFX allocation, it holds that every agent i does not envy a different agent j after the removal of any item from j ’s bundle. The definition of SD-EFX (with respect to EFX) is analogous to the definition of SD-EF (with respect to EF).

⁸In Appendix C we show that SD-EFX allocations do not exist for additive valuations. In contrast, the existence of EFX allocation remains an elusive open problem.

at the beginning of the process, SD-EF fails if any other agent picks i 's most preferred item; towards the end, there is a risk that all the remaining items are bad. Give-Away Round Robin gains by ensuring that every other agent receives one of i 's worst items, making sure that agent i does not receive her least preferred items. At the same time, the analysis becomes significantly trickier. For Round-Robin we could be certain that, other than the last item, items picked by agent i are “good” in the sense that they were the best in some pool of items. For Give-Away Round-Robin, each agent essentially starts with a small, random set of items. Therefore, arguing that SD-EF does not fail because of, say, the first ten items is not clear at all. In our analysis, we consider various cases (failure because of “high,” “middle,” “low,” and last items), which need delicate, separate handling. In Theorem 2, we show that, when m is divisible by n , Give-Away Round-Robin outputs an SD-EF allocation with probability at least $1 - \tilde{O}_n\left(\frac{1}{m}\right)$.

EF allocations under arbitrary valuations. In Section 4, we proceed to study arbitrarily general valuations for the case of $n = 2$ agents. We impose no constraint on the valuation function: it can be superadditive, non-monotone, or negative for some bundles and positive for others. By picking a valuation function such that bundles of size strictly less than $m/2$ and strictly more than $m/2$ are worthless and noticing that random renaming will not affect this property, we conclude that, if envy-free allocations exist, they must allocate exactly $m/2$ items to each agent (and therefore, m must be even). Next, if the valuation function is indifferent between a set of size $m/2$ and its complement, then an envy-free allocation trivially exists; therefore we can assume that, without loss of generality, a subset of size $m/2$ and its complement have different values for every agent i . Randomly renaming a “preferred” subset of size $m/2$ maps it to a non-preferred set with probability $1/2$. Surprisingly, we find that for an even number of items and arbitrary valuation functions v_1, v_2 , the probability that an envy-free allocation exists after random renaming is $1 - \frac{1}{m/2+1}$ (Theorem 5). This bound is almost tight: there exist instances with additive valuations such that an envy-free allocation doesn't exist with probability $1/m$. Our result is based on the following insight: a valuation function can be represented as an $m/2$ -uniform hypergraph on m vertices. Given this, a random permutation of the item names corresponds to picking a random, isomorphic hypergraph. Whenever the hypergraphs corresponding to the two agent's valuation functions are not identical an envy-free allocation exists — allocate i the bundle corresponding to the hyperedge present in i 's hypergraph and not j 's. Using the orbit-stabilizer theorem, we reduce our question about the existence of envy-free allocations to a question about the number of automorphisms of k -uniform hypergraphs. In Theorem 4, we prove that the number of automorphisms of a k -uniform hypergraph on m vertices is at most $\frac{m!}{m-k+1}$. This is the main technical result used to establish Theorem 5 and may be of independent interest.

1.2 Related Work

Dickerson et al. [DGK⁺14] initiated the study of asymptotic fair division and showed that the welfare maximizing algorithm (allocate each good to the agent with the highest value for it) is envy-free with high probability for $m \in \Omega(n \log n)$ for the case of additive, non-negative valuations. Manurangsi and Suksompong [MS20, MS21] establish tight bounds for the existence of envy-free allocations in this setting: $m \in \Omega(n \log n / \log \log n)$ is a necessary and sufficient condition; similar to our Theorem 1, this bound is achieved by the classic Round-Robin algorithm. Bai and Gözl [BG22] extend these results to the case of independent but non-identical additive agents. Kurokawa et al. [KPW16] and Farhadi et al. [FGH⁺19] show that weaker notions of fairness, namely maximin share fairness, also exist with high probability. For non-additive stochastic valuations, Manurangsi and Suksompong [MS21] and Gan et al. [GSV19] study the so-called *house allocation* problem, where there are m houses, n agents, with $v_{i,j} \sim D_i$ for each agent i and house j , and each agent must be allocated a single house; equivalently, one can think of unit-demand valuations and independently picking a uniformly random ranking of the items for each agent. The former paper shows that an envy-free assignment is likely to exist if $m/n > e + \epsilon$, for any constant $\epsilon > 0$, but unlikely to exist if $m/n < e - \epsilon$.

The existence of fair allocations, and their compatibility with efficiency, has also been studied in dynamic settings with additive valuations [BKPP18]. The algorithms of Dickerson et al. [DGK⁺14] and Bai and

Gölz [BG22] can be readily applied even when items arrive in an online fashion, implying that Pareto efficiency ex-post and envy-freeness with high probability are compatible even in an online setting. The same result was recently shown to be (approximately) true even when the designer doesn't have access to the exact values of the agents [BHP22], or even when agents' valuations are correlated [ZP20, BKP⁺22].

Bouveret et al. [BEL⁺10] study preferences represented by SCI-nets, which take the form of a partial order over bundles induced by a linear ordering over items. Order-consistent valuations are exactly those where pairwise dominance of bundles implies envy-freeness. As opposed to our interest here, [BEL⁺10] are interested in computational aspects of the problem (e.g., the complexity of computing an envy-free allocation in the worst-case).

A related research direction is smoothed analysis. Traditionally, in smooth analysis, an instance is generated by starting from a worst-case instance and adding a small amount of noise. This model has been widely successful in circumventing computational hardness [ST04, BS95, FK98, KMM11]. More recently, this model has been used to bypass impossibility results in social choice [Xia20, Xia21, FHP22, Xia21], mechanism design [PSW19, BG21, BCH⁺17], and, closer to this paper, fair division [BFGP22]. Specifically, Bai et al. [BFGP22] start from a worst-case instance for additive agents and add an independent boost to the value of each item for each agent; the authors give sufficient conditions (on the boosting) for envy-free allocations to exist with high probability. One could interpret our model as a smoothed/semi-random model which starts from a worst-case instance and adds noise in the form of random renaming. In contrast with existing models, we permute the items and leave the valuation function untouched.

Further afield, motivated by the non-existence of envy-free allocations in the worst case, a growing literature in fair division studies relaxed versions of envy-freeness. For example, an allocation is envy-free up to one good (EF1) [Bud11] if every agent i prefers her bundle to the bundle of any other agent j after the removal of some item (from j 's bundle). Such allocations are known to exist in the worst-case, even for monotone and non-negative valuations [LMMS04]. For additive valuations, there are allocations that are EF1 and Pareto efficient, but not for subadditive or supermodular valuations [CKM⁺19]. A different relaxation of envy-freeness is envy-free up to any good (EFX) [CKM⁺19], which states that every agent i prefers her bundle to the bundle of any other agent j after the removal of *any* item (from j 's bundle). Such allocations are known to exist for the case of two agents with general valuations [PR20], identical agents [PR20], and three additive agents [CGM20]; a major open problem is whether EFX allocations exist for a general number of (even additive) agents.

2 Preliminaries

We consider the problem of dividing a set \mathcal{M} of m indivisible items among a set \mathcal{N} of $n \geq 2$ agents. Throughout, we assume $\mathcal{N} = \{1, 2, \dots, n\}$ and the items are indexed by $1, 2, \dots, m$. An allocation $\mathcal{A} = (A_1, A_2, \dots, A_n)$ is an n -partition of the set of items \mathcal{M} , where A_i denotes the set of items allocated to agent $i \in \mathcal{N}$. That is, in any allocation \mathcal{A} , all items are allocated, $\bigcup_{i \in \mathcal{N}} A_i = \mathcal{M}$, and each item is allocated to only one agent, $A_i \cap A_j = \emptyset$ for all distinct agents $i, j \in \mathcal{N}$. We sometimes refer to a subset of items as a *bundle*.

Each agent $i \in \mathcal{N}$ has a valuation function $v_i : 2^{\mathcal{M}} \mapsto \mathbb{R}$, which maps each subset of items $S \subseteq \mathcal{M}$ to $v_i(S)$, the agent's value for S . We are interested in proving the existence of allocations that are fair with respect to the agent valuations. Our primary notion of fairness is envy-freeness.

Definition 1 (Envy-freeness). *An allocation $\mathcal{A} = (A_1, \dots, A_n)$ is envy-free (EF) iff each agent prefers her own bundle over the bundle of any other agent, i.e., for all agents $i, j \in \mathcal{N}$, we have $v_i(A_i) \geq v_i(A_j)$.*

For the case of indivisible items, envy-freeness can be too stringent of a requirement. A common relaxation is called envy-freeness up to any good, or EFX.

Definition 2 (Envy-freeness up to any good (EFX)). *An allocation \mathcal{A} is envy-free up to any good (EFX) iff for all agents $i, j \in \mathcal{N}$ where $A_j \neq \emptyset$, we have $v(A_i) \geq v(A_j \setminus \{g\})$ for all items $g \in A_j$.*

In Section 3, we focus on a class of valuation functions that are consistent with an underlying preference order over the items. Given an ordering of the items $\pi = g_1 \succ g_2 \succ \dots \succ g_m$, we can define a partial order over bundles \succeq_π^{sd} as follows. Let $A, \hat{A} \subseteq \mathcal{M}$ be any two bundles such that $A = \{g_{\sigma_1}, g_{\sigma_2}, \dots, g_{\sigma_{|A|}}\}$ and $\hat{A} = \{g_{\hat{\sigma}_1}, g_{\hat{\sigma}_2}, \dots, g_{\hat{\sigma}_{|\hat{A}|}}\}$ where sequences σ and $\hat{\sigma}$ are sorted in an increasing order. We say that $A \succeq_\pi^{\text{sd}} \hat{A}$ iff $|A| \geq |\hat{A}|$ and for all $k \in \{1, 2, \dots, |\hat{A}|\}$, we have $\sigma_k \leq \hat{\sigma}_k$. Additionally, $A \succ_\pi^{\text{sd}} \hat{A}$ iff $A \succeq_\pi^{\text{sd}} \hat{A}$ and $A \neq \hat{A}$.

Definition 3 (Order-consistency). *A valuation function $v : 2^{\mathcal{M}} \mapsto \mathbb{R}$ is called order-consistent with respect to an ordering π iff for all bundles $A, B \subseteq \mathcal{M}$, $A \succeq_\pi^{\text{sd}} B$ implies that $v(A) \geq v(B)$. If v is order consistent with respect to some order π , we will simply call v order-consistent.*

Definition 4 (SD-EF). *Suppose that each agent $i \in \mathcal{N}$ has a valuation function v_i which is order-consistent with respect to π_i . An allocation \mathcal{A} is sd-envy-free (SD-EF) iff for all $i, j \in \mathcal{N}$ we have $A_i \succeq_{\pi_i}^{\text{sd}} A_j$.*

SD-EF allocations are also EF. In fact, an allocation \mathcal{A} which is SD-EF for agents with valuation functions v_1, v_2, \dots, v_n that are order-consistent with respect to $\pi_1, \pi_2, \dots, \pi_n$, respectively, is EF for agents all valuation functions $\hat{v}_1, \hat{v}_2, \dots, \hat{v}_n$ that are order-consistent with respect to $\pi_1, \pi_2, \dots, \pi_n$ respectively. When an allocation $\mathcal{A} = (A_1, A_2, \dots, A_n)$ is SD-EF, it follows that $|A_i| = |A_j|$ for all $i, j \in \mathcal{N}$, i.e., the number of items are a multiple of the number of agents, $m = qn$ for some integer $q \in \mathbb{N}$. When $m \neq qn$ (for $q \in \mathbb{N}$), we consider the analogous strengthening of EFX.

Definition 5 (SD-EFX). *Suppose each agent $i \in \mathcal{N}$ has a valuation function v_i that is order-consistent with respect to π_i . An allocation \mathcal{A} is sd-envy-free up to any item (SD-EFX) iff for all agents $i, j \in \mathcal{N}$, where $A_j \neq \emptyset$, we have $A_i \succeq_{\pi_i}^{\text{sd}} A_j \setminus \{g\}$ for all items $g \in A_j$.*

As with SD-EF and EF, an allocation \mathcal{A} which is SD-EFX is also \mathcal{A} is EFX. While the existence of EFX remains unknown, in Appendix C, we give an instance for which SD-EFX allocations do not exist.

Finally, if v is a valuation function and π is a permutation over items, we will use the notation v^π to represent the permuted valuation function after random renaming, where $v^\pi(S) := v(\pi^{-1}(S))$, and $\pi(S) := \{\pi(g) : g \in S\}$. If v is an order-consistent valuation, then v^π is order-consistent with respect to π .

3 Order-Consistent Valuations

Our first main result is that SD-EF allocations exist with high probability for order-consistent valuation functions. In fact, the well-known Round-Robin algorithm, presented as Algorithm 1, outputs such allocations (with high probability) in polynomial time. Round-Robin takes as input an ordering of the items for each agent, and agents take turns taking the best available remaining item according to this order. As long as Round-Robin has access to these orderings after renaming (that is, no other information about the valuation functions is needed), and as long as m is divisible by n , Round-Robin will result in an SD-EF allocation with high probability. The precise probability is asymptotically optimal. Beyond this, if m is not divisible by n (so there is no hope for an EF allocation), Round-Robin will find an SD-EFX allocation with, once again, asymptotically optimal probability.

Theorem 1. *Let π_1, \dots, π_n be sampled independently and uniformly at random. When $m = qn$, $q \in \mathbb{N}$,*

$$\Pr[\text{Round Robin is SD-EF}] \geq 1 - O\left(\frac{n^2}{m} + \frac{n \log m}{m^{\frac{n-1}{n}}}\right).$$

Further, for any m ,

$$\Pr[\text{Round Robin is SD-EFX}] \geq 1 - O\left(\frac{n^2}{m}\right),$$

and, if $m \geq 2n$,

$$\Pr[\text{No SD-EFX allocation exists}] \in \Omega\left(\min\left(\frac{n^2}{m}, 1\right)\right).$$

Algorithm 1: Round Robin Algorithm

Input : Valuations v_1, \dots, v_n which are order consistent with respect to π_1, \dots, π_n .
Output: Allocation $\mathcal{A} = (A_1, A_2, \dots, A_n)$

Set $A_i \leftarrow \emptyset$ for all agents $i \in \mathcal{N}$
Set $P \leftarrow \mathcal{M}$
for $i = 1, \dots, n, 1, \dots, n, 1, \dots$ **do**
 Let $g \in P$ be the unallocated item with the lowest index in π_i .
 Set $A_i \leftarrow A_i \cup \{g\}$
 Set $P \leftarrow P \setminus \{g\}$
return (A_1, A_2, \dots, A_n)

Notice that if $m \in \omega(n^2)$, then, as n grows, SD-EF/SD-EFX allocations exist with high probability. The lower bound on the non-existence of SD-EFX implies that when $m \in O(n^2)$, no SD-EF (or even EF) allocation exists. In this sense, Round-Robin is asymptotically optimal.

For ease of exposition, we break the proof of Theorem 1 into separate lemmas. We first assume that $m = qn$ for some integer q , and prove the existence of SD-EF allocations.

Lemma 1 (Round-Robin is SD-EF). *When $m = qn$, $\Pr[\text{Round Robin is SD-EF}] \geq 1 - O\left(\frac{n^2}{m} + \frac{n \log m}{m^{\frac{n-1}{n}}}\right)$.*

Second, we show how to extend our arguments to the case of arbitrary m and show the existence of SD-EFX allocations.

Lemma 2 (Round-Robin is SD-EFX). *For any m , $\Pr[\text{Round Robin is SD-EFX}] \geq 1 - O\left(\frac{n^2}{m}\right)$.*

Finally, we show the lower bound on the probability that SD-EFX allocations exist.

Lemma 3. *For any m , if $m \geq 2n$, $\Pr[\text{No SD-EFX allocation exists}] \in \Omega\left(\min\left(\frac{n^2}{m}, 1\right)\right)$.*

The three lemmas combined imply Theorem 1. We prove Lemma 1 in Section 3.1. Lemma 2 largely follows the same proof; we prove it in Section 3.2. Finally, Lemma 3 is proved in Section 3.3.

In Section 3.4, we observe that, even though Round-Robin is asymptotically optimal as n grows, for the important case of small (i.e., constant) n , it leaves a small gap in our understanding. We introduce a new algorithm, Give-Away Round Robin, which closes this gap.

3.1 Round-Robin is SD-EF: The proof of Lemma 1

Our goal will be to show that the probability that Round-Robin does not output an SD-EF allocation is at most $\frac{33n^2}{m} + \frac{16n \log m}{m^{1-1/n}}$. We restrict our analysis to m and n such that $m \geq 2n$ and $\frac{n \log m}{m^{n-1/n}} \leq 1/16$, since otherwise, our target upper bound holds trivially. Fix m , n , and q such that $m = qn$. In each round of the Round-Robin algorithm, lower-indexed agents pick before higher-indexed ones, and therefore the former do not envy the latter. That is, agent i can only envy agent j 's bundle if $j < i$. For agents $i, j \in \mathcal{N}$ with $j < i$, let \mathcal{E}^{ij} be the event that $A_i \not\prec_{\pi_i}^{\text{sd}} A_j$, i.e., agent i does not sd-prefer their own bundle over that of agent j . The probability that the final allocation is not SD-EF is then exactly the probability that one of these events occurs; formally, $\Pr\left[\bigcup_{i \in \mathcal{N}} \bigcup_{j: j < i} \mathcal{E}^{ij}\right]$.

We use $g \succ_i g'$ to denote that item g is preferred to item g' under π_i . For a set of items $S \subseteq \mathcal{M}$, we write $g \succ_i S$ when g is preferred to all items in S , i.e., $g \succ_i g'$ for all $g' \in S$. Additionally, we use P_t to denote the pool of available items at “time” t , i.e., after t picks have been made (so, $P_0 = \mathcal{M}$ and $P_m = \emptyset$). Let g_t be the t 'th item picked during the execution of the Round Robin algorithm, that is, $P_{t+1} = P_t \setminus \{g_{t+1}\}$. Since agents pick items sequentially as per their index, g_t is picked by agent j exactly when $t \equiv j \pmod n$, $g_{(\ell-1)n+j}$ is the ℓ 'th item picked by agent j , and $A_j = \{g_{0 \cdot n+j}, \dots, g_{(q-1)n+j}\}$.

Taking the perspective of agent i , let b_{jk}^i denote the k 'th best item according to π_i in the final bundle A_j ; we will write b_{jk} if the agent i is clear from the context. For agent i 's picks, $b_{ik} = g_{(k-1)n+i}$, i.e., their k 'th pick is also their k 'th favorite item in their bundle. However, this is not the case for items in others' bundles. Since the final bundles are all of size q , $A_i \succeq_{\pi_i}^{\text{sd}} A_j$ exactly when $b_{ik} \succ_i b_{jk}$ for all $k \leq q$. Hence, we decompose the event \mathcal{E}^{ij} into events \mathcal{E}_k^{ij} , where \mathcal{E}_k^{ij} is the event that $b_{jk} \succ_i b_{ik}$. Now $\mathcal{E}^{ij} = \bigcup_{k=1}^q \mathcal{E}_k^{ij}$, and (by a union bound) our goal will be to upper bound

$$\sum_{i \in \mathcal{N}} \sum_{k=1}^q \Pr \left[\bigcup_{j: j < i} \mathcal{E}_k^{ij} \right]. \quad (1)$$

Fix an agent i and value k . To get a handle on $\Pr \left[\bigcup_{j \in \mathcal{N}: j < i} \mathcal{E}_k^{ij} \right]$, we condition on agent i 's ordering of the items, π_i . Whenever agent i picks g_t , g_t must be the best item remaining in P_{t-1} according to π_i . When agent $j \neq i$ picks g_t (so $t \not\equiv i \pmod{n}$), the distribution of g_t over the random draws of $\pi_{-i} = \{\pi_1, \dots, \pi_n\} \setminus \{\pi_i\}$, even conditioned on all previous picks g_1, \dots, g_{t-1} , is uniformly random over P_{t-1} . Let R_ℓ be the set of items picked between agent i 's ℓ 'th and $(\ell+1)$ 'th pick. That is, $R_0 = \{g_1, \dots, g_{i-1}\}$, for $1 \leq \ell \leq q-1$, $R_\ell = \{g_{n(\ell-1)+i+1}, \dots, g_{\ell n+i-1}\}$, and $R_q = \{g_{n(q-1)+i+1}, \dots, g_m\}$. From the perspective of agent i , each item in R_ℓ is picked sequentially and uniformly at random from the remaining items. This allows us to analyze an equivalent two-step process wherein (1) R_ℓ is first selected uniformly at random from P_t , after which, (2) each item in R_ℓ is matched to a uniformly random agent (different than i) that picked in the corresponding timesteps.

We upper bound the expression in (1) by upper bounding each of the summands. Our analysis splits into cases based on the value of k , as care must be taken when comparing i 's worst item to that of other agents.

Lemma 4. *For $k < q$, it holds that $\Pr \left[\bigcup_{j: j < i} \mathcal{E}_k^{ij} \right] \leq \frac{8}{\binom{q}{k}}$.*

Lemma 5. *For $k = q$, it holds that $\Pr \left[\bigcup_{j: j < i} \mathcal{E}_q^{ij} \right] \leq \frac{n}{m} + 2 \left(\frac{8n \log m}{m^{1-1/n}} \right)^{n+1-i}$.*

We show how to conclude the proof of Lemma 1 given these two lemmas, then proceed to prove them. Using Lemmas 4 and 5, we have

$$\begin{aligned} \Pr \left[\bigcup_{i,j} \mathcal{E}^{ij} \right] &\leq \sum_{i \in \mathcal{N}} \left(\sum_{k=1}^{q-1} \Pr \left[\bigcup_j \mathcal{E}_k^{ij} \right] + \Pr \left[\bigcup_j \mathcal{E}_q^{ij} \right] \right) \\ &\leq \sum_{i \in \mathcal{N}} \left(\sum_{k=1}^{q-1} \frac{8}{\binom{q}{k}} + \frac{n}{m} + 2 \left(\frac{8n \log m}{m^{1-1/n}} \right)^{n+1-i} \right) \\ &= 8n \sum_{k=1}^{q-1} \frac{1}{\binom{q}{k}} + \frac{n^2}{m} + 2 \sum_{i=1}^n \left(\frac{8n \log m}{m^{1-1/n}} \right)^{n+1-i}. \end{aligned}$$

We now bound each of these sums. For the first,

$$\sum_{k=1}^{q-1} \frac{1}{\binom{q}{k}} \leq \frac{1}{\binom{q}{1}} + \frac{1}{\binom{q}{q-1}} + \sum_{k=2}^{q-2} \frac{1}{\binom{q}{k}} \leq \frac{2}{q} + \sum_{k=2}^{q-2} \frac{1}{\binom{q}{2}} = \frac{2}{q} + \frac{2(q-3)}{q(q-1)} \leq \frac{4}{q} = \frac{4n}{m}. \quad (2)$$

For the second, since $\frac{8n \log m}{m^{1-1/n}} \leq 8/16 = 1/2$,

$$\sum_{i=1}^n \left(\frac{8n \log m}{m^{1-1/n}} \right)^{n+1-i} = \sum_{i=1}^n \left(\frac{8n \log m}{m^{1-1/n}} \right)^i \leq \sum_{i=1}^{\infty} \left(\frac{8n \log m}{m^{1-1/n}} \right)^i = \frac{\frac{8n \log m}{m^{1-1/n}}}{1 - \frac{8n \log m}{m^{1-1/n}}} \leq \frac{16n \log m}{m^{1-1/n}}.$$

We conclude that

$$\Pr \left[\bigcup_{i,j} \mathcal{E}^{ij} \right] \leq \frac{33n^2}{m} + \frac{16n \log m}{m^{1-1/n}} \in O \left(\frac{n^2}{m} + \frac{n \log m}{m^{\frac{n-1}{n}}} \right),$$

as needed.

Proof of Lemma 4. We will directly upper bound each $\Pr \left[\mathcal{E}_k^{ij} \right]$ and union bound over the (at most) n possible choices of j . The equivalent random process of picking R^ℓ and assigning each item randomly discussed above implies $\Pr \left[\mathcal{E}_k^{ij} \right] = \Pr \left[\mathcal{E}_k^{ij'} \right]$, for each $j, j' < i$. Indeed, the items assigned to j and j' in this process appear in exactly the same R_ℓ sets. Now the distributions of A_j and $A_{j'}$ are identical even when conditioning on A_i . As a result, we only upper bound $\Pr \left[\mathcal{E}_k^{ij} \right]$ for $j = i - 1$; the same bound holds for all agents that i could possibly envy (i.e., for all agents j' with $j' < i$).

Consider the item $g_{(k-1)n+i}$, which is picked by i in the k 'th round. As noted above, $g_{(k-1)n+i} = b_{ik}$, so \mathcal{E}_k^{ij} occurs exactly when $b_{jk} \succ_i b_{ik} = g_{(k-1)n+i}$. This can only occur if there are k timesteps, t_1, \dots, t_k , such that j picks an item g_{t_z} satisfying $g_{t_z} \succ_i g_{(k-1)n+i}$. For any $t > (k-1)n+i$, $g_{(k-1)n+i} \succ_i g_t$, since i always picks the best available remaining item. Hence, all k of these (unfortunate) picks must have occurred before i 's k 'th pick. Agent j made exactly k picks before i made their k 'th pick, so for $b_{jk} \succ_i g_{(k-1)n+i}$ to hold, a necessary and sufficient condition is that all these picks were preferred to $g_{(k-1)n+i}$ (according to π_i), i.e., $g_{(\ell-1)n+j} \succ_i g_{(k-1)n+i}$ for all $\ell = 1, \dots, k$.

Let $P_t^{(r)}$ be the r 'th best item remaining in P_t according to π_i . For all $t < t'$, $|P_t \setminus P_{t'}| = t' - t$, i.e., there are $t' - t$ fewer items available after $t' - t$ picks. This implies that for all $t < (k-1)n+i$, $g_{(k-1)n+i} \succeq_i P_t^{((k-1)n+i-t)}$.⁹ Furthermore, $g_{(k-1)n+i} = P_{(k-1)n+i-1}^{(1)}$ as i picks the best available item remaining at time $(k-1)n+i-1$. $|P_t \setminus P_{((k-1)n+i-1)}| = (k-1)n+i-t-1$, so at least one of $P_t^{(1)}, \dots, P_t^{((k-1)n+i-t)}$ must be available at time $(k-1)n+i-1$, and the best available item at time $(k-1)n+i-1$ must be at least as good as the worst of these. This implies that if $g_{(\ell-1)n+j} \succ_i g_{(k-1)n+i}$, then $g_{(\ell-1)n+j} \succ_i P_{(\ell-1)n+j-1}^{((k-\ell)n+(i-j)+1)}$. Using the fact that $j = i - 1$, this simplifies to $g_{(\ell-1)n+j} \succeq_i P_{(\ell-1)n+j-1}^{((k-\ell)n+1)}$ or, in words, agent j 's ℓ 'th pick must be one of the top $(k-\ell)n+1$ available items according to π_i . This necessary condition means that

$$\Pr \left[\mathcal{E}_k^{ij} \right] \leq \Pr \left[\bigcap_{\ell=1}^k \left\{ g_{(\ell-1)n+j} \succeq_i P_{(\ell-1)n+j-1}^{((k-\ell)n+1)} \right\} \right].$$

The key observation is that each of the events $g_{(\ell-1)n+j} \succeq_i P_{(\ell-1)n+j-1}^{((k-\ell)n+1)}$, over the samples of π_{-i} , are mutually independent. Hence, we get

$$\Pr \left[\bigcap_{\ell=1}^k \left\{ g_{(\ell-1)n+j} \succeq_i P_{(\ell-1)n+j-1}^{((k-\ell)n+1)} \right\} \right] = \prod_{\ell=1}^k \Pr \left[g_{(\ell-1)n+j} \succeq_i P_{(\ell-1)n+j-1}^{((k-\ell)n+1)} \right].$$

Furthermore, $g_{(\ell-1)n+j}$ is a uniform random sample from $P_{(\ell-1)n+j-1}$, so

$$\Pr \left[g_{(\ell-1)n+j} \succeq_i P_{(\ell-1)n+j-1}^{((k-\ell)n+1)} \right] = \frac{(k-\ell)n+1}{|P_{(\ell-1)n+j-1}|}.$$

Noticing that $|P_t| = m - t$, we get that,

$$\Pr \left[\mathcal{E}_k^{ij} \right] = \prod_{\ell=1}^k \frac{(k-\ell)n+1}{|P_{(\ell-1)n+j-1}|} = \prod_{\ell=1}^k \frac{(k-\ell)n+1}{m - ((\ell-1)n+j-1)} = \prod_{\ell=1}^k \frac{(k-\ell)n+1}{(q-\ell)n+n-j+1} \quad (3)$$

⁹And this is tight only when all picks $g_t, \dots, g_{(k-1)n+i-1}$ are preferred to $g_{(k-1)n+i}$ in π_i .

$$\leq \prod_{\ell=1}^k \frac{(k-\ell)n+1}{(q-\ell)n+1} = \prod_{\ell=1}^k \frac{k-\ell+1/n}{q-\ell+1/n} = \frac{\prod_{\ell=0}^{k-1} (1/n + \ell)}{\prod_{\ell=0}^{k-1} (q-k+1/n + \ell)} = \frac{(1/n)^{\bar{k}}}{(q-k+1/n)^{\bar{k}}}.$$

The inequality follows because $(n-j) \geq 0$, and the second to last equality follows by a change of variables. The notation $(x)^{\bar{r}}$ here represents the rising factorial $(x)^{\bar{r}} = x(x+1)\cdots(x+r-1)$. We make use of the known equality $(x)^{\bar{r}} = \frac{\Gamma(x+r)}{\Gamma(x)}$ to get that

$$\Pr \left[\mathcal{E}_k^{ij} \right] = \frac{\Gamma(k+1/n)}{\Gamma(1/n)} \cdot \frac{\Gamma(q-k+1/n)}{\Gamma(q+1/n)} \leq \frac{2}{n} \cdot \frac{\Gamma(k+1/n)\Gamma(q-k+1/n)}{\Gamma(q+1/n)},$$

where we replaced $\Gamma(1/n)$ by using the fact that $\Gamma(1+x) = x\Gamma(x)$ for all x (which implies that $\Gamma(1/n) = n\Gamma(1+1/n)$), and that the Γ function has a global minimum on the positive reals above $1/2$ [Wre68] (and thus, $n\Gamma(1+1/n) \geq \frac{n}{2}$).

Ideally, we could replace the $1/n$ terms inside the gamma function with 1s, which would allow us to convert the Gammas to the corresponding factorial terms, and would simplify to $1/\binom{q}{k}$. We show that this is approximately correct by making use of Gautchi's inequality [Gau59], which states

$$x^{1-s} \leq \frac{\Gamma(x+1)}{\Gamma(x+s)} \leq (x+1)^{1-s},$$

for all $x > 0$ and $s \in (0, 1)$. We state our bound as the following claim, the proof of which is deferred to Appendix B.

Claim 1. $\frac{2}{n} \cdot \frac{\Gamma(k+1/n)\Gamma(q-k+1/n)}{\Gamma(q+1/n)} \leq \frac{8}{n} \cdot \frac{1}{\binom{q}{k}}$.

Therefore, for all $j < i$ and $k < q$, $\Pr \left[\mathcal{E}_k^{ij} \right] \leq \frac{8}{n} \cdot \frac{1}{\binom{q}{k}}$. Union bounding over the (at most) n possible values of j , we have that

$$\Pr \left[\bigcup_{j:j < i} \mathcal{E}_k^{ij} \right] \leq \frac{8}{\binom{q}{k}}. \quad (4)$$

□

Proof of Lemma 5. Let $L \subseteq \mathcal{M}$ be the set of the bottom $\lceil 3n \log m \rceil + (n-1)$ items according to π_i . Notice that $\lceil 3n \log m \rceil + (n-1) \leq 3n \log m + n \leq 4n \log m$ since $\log(m) \geq \log(2n) \geq \log(4) \geq 1$.¹⁰ Recall that b_{iq} is the worst item in agent i 's bundle. We consider the event that $b_{iq} \notin L$. We show that the probability of $b_{iq} \in L$ is small and, conditioned on $b_{iq} \notin L$, it is unlikely that i envies anyone. Formally, notice that

$$\begin{aligned} \Pr \left[\bigcup_{j:j < i} \mathcal{E}_q^{ij} \right] &= \Pr \left[\bigcup_{j:j < i} \mathcal{E}_q^{ij} \mid b_{iq} \notin L \right] \cdot \Pr [b_{iq} \notin L] + \Pr \left[\bigcup_{j:j < i} \mathcal{E}_q^{ij} \mid b_{iq} \in L \right] \cdot \Pr [b_{iq} \in L] \\ &\leq \Pr \left[\bigcup_{j:j < i} \mathcal{E}_q^{ij} \mid b_{iq} \notin L \right] + \Pr [b_{iq} \in L] \\ &\leq \sum_{j:j < i} \Pr [\mathcal{E}_q^{ij} \mid b_{iq} \notin L] + \Pr [b_{iq} \in L]. \end{aligned}$$

We upper bound each of these terms individually.

Claim 2. For all $j < i$, $\Pr [\mathcal{E}_q^{ij} \mid b_{iq} \notin L] \leq 1/m$.

Claim 3. $\Pr [b_{iq} \in L] \leq 2 \left(\frac{8n \log m}{m^{1-1/n}} \right)^{n+1-i}$.

¹⁰We always interpret \log as the natural log.

Given these two bounds, we can conclude the proof of the lemma as follows:

$$\Pr \left[\bigcup_{j:j<i} \mathcal{E}_q^{ij} \right] \leq \sum_{j:j<i} \Pr [\mathcal{E}_q^{ij} \mid b_{iq} \notin L] + \Pr [b_{iq} \in L] \leq \frac{n}{m} + 2 \left(\frac{8n \log m}{m^{1-1/n}} \right)^{n+1-i}. \quad (5)$$

It remains to prove the two claims.

Proof of Claim 2. Notice that $b_{iq} \notin L$ implies $A_i \cap L = \emptyset$, so all the items in L must have been chosen by other agents. If $A_j \cap L \neq \emptyset$ it follows that $b_{jq} \in L$ and, hence, $b_{iq} \succ_i b_{jq}$. Therefore, $\Pr [\mathcal{E}_q^{ij} \mid b_{iq} \notin L] \leq \Pr [A_j \cap L = \emptyset \mid b_{iq} \notin L]$ and it suffices to upper bound the latter.

To analyze the bundle of an agent $j \neq i$, we revert to the second view of the allocation process in which the bundles R_0, \dots, R_q are first sampled and then matched to agents different that i . We condition on any valid R_0, \dots, R_q such that $L \subseteq \bigcup_{\ell} R_{\ell}$, and now consider sampling the actual bundles $A_{j'}$ for $j' \neq i$ from these. Except for R_0 and R_q , each $|R_{\ell}| = n - 1$ and A_j contains exactly one item from R_{ℓ} chosen uniformly at random. Therefore, with probability $\frac{|R_{\ell} \cap L|}{n-1}$, $A_j \cap R_{\ell} \cap L \neq \emptyset$. In words, if there are r items from L in R_{ℓ} , then j receives one of them with probability $r/(n-1)$. Importantly, these are independent across rounds ℓ . It follows that

$$\Pr [A_j \cap L = \emptyset \mid b_{iq} \notin L] \leq \Pr \left[\sum_{\ell=1}^{q-1} \mathbb{I}[A_j \cap L \cap R_{\ell} = \emptyset] = 0 \right].$$

By linearity of expectation and the choice of L ,

$$\mathbb{E} \left[\sum_{\ell=1}^{q-1} \mathbb{I}[A_j \cap L \cap R_{\ell}] \right] = \sum_{\ell=1}^{q-1} \frac{|R_{\ell} \cap L|}{n-1} \geq \frac{|L| - |R_0| - |R_q|}{n-1} \geq \frac{|L| - (n-1)}{n-1} \geq \frac{3n \log m}{n-1} \geq 3 \log m. \quad (6)$$

The sum $\sum_{\ell=1}^{q-1} \mathbb{I}[A_j \cap L \cap R_{\ell} = \emptyset]$ is the sum of independent Bernoulli variables indicating whether j received an item from L in round ℓ . Applying a Chernoff bound we have

$$\begin{aligned} \Pr \left[\sum_{\ell=1}^{q-1} \mathbb{I}[A_j \cap L \cap R_{\ell} = \emptyset] = 0 \right] &\leq \Pr \left[\sum_{\ell=1}^{q-1} \mathbb{I}[A_j \cap L \cap R_{\ell} = \emptyset] < \left(1 - \sqrt{\frac{2}{3}} \right) \mathbb{E} \left[\sum_{\ell=1}^{q-1} \mathbb{I}[A_j \cap L \cap R_{\ell}] \right] \right] \\ &\leq \exp \left(\frac{- \left(\sqrt{\frac{2}{3}} \right)^2 \mathbb{E} \left[\sum_{\ell=1}^{q-1} \mathbb{I}[A_j \cap L \cap R_{\ell}] \right]}{2} \right) \\ &\leq \exp \left(\frac{- \frac{2}{3} (3 \log m)}{2} \right) \\ &= \exp(-\log m) = \frac{1}{m}. \end{aligned}$$

Putting this all together, we have that

$$\Pr [\mathcal{E}_q^{ij} \mid b_{iq} \notin L] \leq \Pr [A_j \cap L = \emptyset \mid b_{iq} \notin L] \leq 1/m, \quad (7)$$

which concludes the proof of Claim 2. \square

The proof of Claim 3 is deferred to Appendix B. This concludes the proof of Lemma 5. \square

3.2 Round-Robin is SD-EFX: The proof of Lemma 2

We extend the analysis of Section 3.1 to get SD-EFX allocations for arbitrary m . Fix an arbitrary m and let $m = qn + r$ for $1 \leq r \leq n$ and $r, q \in \mathbb{N}$. Let \mathcal{A}^q be the allocation after qn steps, and \mathcal{A} be the complete allocation of all $qn + r$ items. Also, let P_t denote the pool of available items at “time” t , i.e., after t picks have been made (so, $P_0 = \mathcal{M}$ and $P_m = \emptyset$). Notice that

$$\begin{aligned} \Pr[\mathcal{A} \text{ is SD-EFX}] &\geq \Pr[\mathcal{A} \text{ is SD-EFX and } \mathcal{A}^q \text{ is SD-EF}] \\ &= \Pr[\mathcal{A} \text{ is SD-EFX} \mid \mathcal{A}^q \text{ is SD-EF}] \cdot \Pr[\mathcal{A}^q \text{ is SD-EF}]. \end{aligned}$$

We show that $\Pr[\mathcal{A} \text{ is SD-EFX} \mid \mathcal{A}^q \text{ is SD-EF}] = 1$ and $\Pr[\mathcal{A}^q \text{ is SD-EF}] \geq 1 - O\left(\frac{n^2}{m}\right)$.

We begin with the former. Fix arbitrary samples π_1, \dots, π_n such that \mathcal{A}^q is SD-EF. We want to show that \mathcal{A} is SD-EFX. Consider agents $i, j \in \mathcal{N}$. We show that either $A_i^q \succeq_{\pi_i}^{\text{sd}} A_j$ or, for all $g \in A_j$, $A_i^q \succeq_{\pi_i}^{\text{sd}} A_j \setminus \{g\}$. Since $A_i^q \subseteq A_i$, the same holds for A_i . It must be that either $A_j = A_j^q$ or $A_j = A_j^q \cup \{g\}$ for some $g \in P_{qn}$. If $A_j = A_j^q$ it follows that $A_i^q \succeq A_j^q = A_j$, since \mathcal{A}^q is SD-EF by assumption. Suppose instead that $A_j = A_j^q \cup \{g^*\}$ for some $g^* \in P_{qn}$. The key observation is that $g^* \in P_{qn}$, implying it was available (and not selected) at every time step that i picked an item, so $g' \succ g^*$ for all $g' \in A_i$. Note that, this doesn't immediately imply the EFX condition, since g^* might not be the worst item (from i 's perspective) that j owns. Let b_{ik} and b_{jk} denote the k 'th best item in A_i^q and A_j^q according to π_i , respectively. By assumption $b_{ik} \succ_i b_{jk}$ for all k . Let b'_{jk} be the k 'th best item in $A_j \setminus \{g^*\}$. If $g^* = g$ then $A_i^q \succeq_{\pi_i}^{\text{sd}} A_j \setminus \{g^*\} = A_j^q$. Otherwise, if $g^* \neq g$, suppose $g^* = b'_{j\ell}$ for some ℓ . Notice that for all $k < \ell$, $b'_{jk} = b_{jk}$ so $b_{ik} \succ_i b'_{jk}$. For $k \geq \ell$, we have that $b_{ik} \succ_i g^* = b_{j\ell} \succeq_i b'_{jk}$. Hence, $A_j^q \succeq_{\pi_i}^{\text{sd}} A_j \setminus \{g^*\}$, as needed.

Next, we show $\Pr[\mathcal{A}^q \text{ is SD-EF}] \geq 1 - O\left(\frac{n^2}{m}\right)$. Specifically, we will show that the probability that \mathcal{A}^q is not SD-EF is upper bounded by $\frac{2066n^2}{m}$. We assume that $m \geq 2000n^2$ (otherwise the bound is trivial), which implies that $qn \geq m/2$. The analysis is very similar to showing Round Robin is SD-EF when $m = qn$, so we simply describe which changes need to be made. Let b_{jk}^i denote the k 'th best item according to π_i in the bundle A_j^q ; for ease of notation we write b_{jk} when i is clear from the context. Let \mathcal{E}_k^{ij} be the that $b_{jk} \succ_i b_{ik}$, and let $\mathcal{E}^{ij} = \bigcup_{k=1}^q \mathcal{E}_k^{ij}$ be $A_i^q \not\succeq_{\pi_i}^{\text{sd}} A_j^q$. That is, \mathcal{E}^{ij} , b_{jk}^i , and \mathcal{E}_k^{ij} are defined analogously to the proof of Lemma 1, but with respect to \mathcal{A}^q instead of \mathcal{A} . Let R_ℓ be the set of items picked between agent i 's ℓ 'th and $(\ell + 1)$ 'th pick, and $R_q = P_{qn}$ be the set of items remaining items after the allocations in \mathcal{A}^q .

On a high level, our goal is again to upper bound by getting bounds on $\Pr\left[\mathcal{E}_k^{ij}\right]$ and $\sum_{i \in \mathcal{N}} \Pr\left[\bigcup_{j:j < i} \mathcal{E}_q^{ij}\right]$, similarly to Lemmas 4 and 5. Using very similar bounds we have

$$\begin{aligned} \Pr\left[\bigcup_{i,j} \mathcal{E}^{ij}\right] &\leq \sum_{i \in \mathcal{N}} \sum_{k=1}^q \Pr\left[\bigcup_{j:j < i} \mathcal{E}_k^{ij}\right] \\ &\leq \sum_{i \in \mathcal{N}} \left(\sum_{k=1}^{q-1} \Pr\left[\bigcup_j \mathcal{E}_k^{ij}\right] + \Pr\left[\bigcup_j \mathcal{E}_q^{ij}\right] \right) \\ &\leq n \sum_{k=1}^{q-1} \Pr\left[\bigcup_j \mathcal{E}_k^{ij}\right] + \sum_{i \in \mathcal{N}} \Pr\left[\bigcup_j \mathcal{E}_q^{ij}\right] \\ &\leq 8n \sum_{k=1}^{q-1} \frac{1}{\binom{q}{k}} + \sum_{i \in \mathcal{N}} \Pr\left[\bigcup_j \mathcal{E}_q^{ij}\right]. \end{aligned}$$

Since $qn \geq m/2$ we can, similarly to Inequality (2), bound the first term by $\frac{64n^2}{m}$. For the second term, one complication is that our bound on $\Pr\left[\bigcup_j \mathcal{E}_q^{ij}\right]$ needs to take care of the $n = 2$ case, separately.

This is caused because the analog of Lemma 5 is not a bound of $\frac{n}{m} + 2 \left(\frac{8n \log m}{m^{1-1/n}} \right)^{n+1-i}$, but a bound of $\frac{n}{m} + 2 \left(\frac{10n \log m}{m^{1-1/n}} \right)^{n+1-i+r}$, which is too weak for $n = 2$. We explain these differences, and how to address them, in Appendix B.1.

3.3 Round-Robin is asymptotically optimal: The proof of Lemma 3

Fix n and m with $m \geq 2n$. We prove that SD-EFX allocations do not exist with probability at least $\min\left(\frac{n^2}{8m}, \frac{1}{8}\right)$. To do so, we first show that if two agents have the same favorite item, no SD-EFX allocation exists. We then show using standard birthday paradox bounds that the probability two agents have the same favorite item is at least $\min\left(\frac{n^2}{8m}, \frac{1}{8}\right)$.

For the former, suppose two agents have the same favorite item g with $m \geq 2n$. We show that no SD-EFX allocation exists. Fix an allocation \mathcal{A} . First note that if \mathcal{A} is not *balanced*, i.e., $||A_i| - |A_j|| \leq 1$, then it cannot be SD-EFX. Indeed, if $|A_i| - |A_j| > 1$, then for any $g \in A_i$, $A_i \setminus \{g\} > |A_j|$ so it cannot be the case that $|A_j| \succeq_{\pi_j}^{\text{sd}} A_i \setminus \{g\}$. Next, suppose \mathcal{A} is balanced. Since the two agents have the same favorite item, there is some agent i whose favorite good is g and who did not receive g . Further, there is an agent j such that $g \in A_j$, and since \mathcal{A} is balanced, $|A_j| \geq 2$. Therefore, there is some $g' \neq g$ such that $g' \in A_j$. However, since $g \in A_j \setminus \{g'\}$, $A_i \not\succeq_{\pi_i}^{\text{sd}} A_j \setminus \{g'\}$, so \mathcal{A} is not SD-EFX.

Next, we show that the probability two agents have the same favorite item is at least $\min\left(\frac{n^2}{8m}, \frac{1}{8}\right)$. Notice that each agent's favorite item is a uniformly selected item, independent of other agents. The probability that two agents have the same first item is known from the collision analysis of Hash functions. Indeed, it is known as long as $n \leq \sqrt{2m}$, the probability that two agents have the same first item is at least $\frac{n(n-1)}{4m}$ [KL20]. In other words, as long as $m \geq n^2/2$, two agents have the same first item with probability at least $\frac{n(n-1)}{4m} \geq \frac{n^2}{8m}$. For $m < n^2/2$, notice that the probability of a collision is decreasing in m . Hence, the probability of a collision is at least the probability with n agents and $m' = \lceil n^2/2 \rceil \leq n^2$ items. The previous result says that this probability is then at least $\frac{n^2}{8m'} \geq \frac{1}{8}$.

3.4 Give-Away Round Robin

As we've shown, Round-Robin is asymptotically optimal as n grows. However, while it is often reasonable to assume that the number of items is large, it is often less reasonable to assume the same for the number of agents. When n is a constant, no SD-EF allocations exist with probability $\Omega_n(1/m)$. Although Round-Robin finds SD-EF allocations with probability $1 - O_n(1/m)$, when m is divisible by n , Theorem 1 only gives an upper bound of $1 - O_n\left(\frac{\log m}{m^{1-1/n}}\right)$ on the probability of finding such an allocation. A careful analysis shows this is relatively tight: with probability $\Omega_n\left(\frac{1}{m^{1-1/n}}\right)$, the last agent will be left with their worst item, which implies the resulting allocation cannot be SD-EF. For example, with $n = 2$, Round-Robin will not find an SD-EF allocation with probability $\Omega(1/\sqrt{m})$, while the lower bound only implies such allocations don't exist with probability $\Omega(1/m)$. This motivates an alternative algorithm that closes this gap. To avoid receiving a truly bad item, agents first *give away* a bad item to each of the other agents, then proceeds with regular Round-Robin on the remaining items. We formalize this in Algorithm 2, and prove it achieves optimal (up to log factors) probability of finding an SD-EF allocation.

Theorem 2. *Let π_1, \dots, π_n be sampled independently and uniformly at random. When $m = qn$, $q \in \mathbb{N}$,*

$$\Pr[\text{Give-Away Round Robin is SD-EF}] \geq 1 - \tilde{O}_n\left(\frac{1}{m}\right).$$

Proof. The proof follows a very similar structure to the proof of Theorem 1, though with more intricate analysis in various places. Fix $m = qn$ for an integer q . We will show the probability that Give-Away Round

Algorithm 2: Give-Away Round Robin

Input: Valuations v_1, \dots, v_n which are order consistent with respect to π_1, \dots, π_n .

Output: An allocation $\mathcal{A} = (A_1, A_2, \dots, A_n)$.

Set $A_i \leftarrow \emptyset$ for all agents $i \in \mathcal{N}$

Set $P \leftarrow \mathcal{M}$

Phase 1: Give-away ($n(n-1)$ items)

for $i = 1, \dots, n$ **do**

for $j = 1, \dots, n, j \neq i$ **do**

Let $g \in P$ be the unallocated item with the highest index in π_i .

Set $A_j \leftarrow A_j \cup \{g\}$

Set $P \leftarrow P \setminus \{g\}$

Phase 2: Round-Robin

for $i = 1, \dots, n, 1, \dots, n, 1, \dots$ **do**

Let $g \in P$ be the unallocated item with the lowest index in π_i .

Set $A_i \leftarrow A_i \cup \{g\}$

Set $P \leftarrow P \setminus \{g\}$

return (A_1, A_2, \dots, A_n)

Robin is not SD-EF is upper bounded by

$$\frac{67584n^3 \log^2 m + 25216n^4 \log m + 2354n^5 + 128n^2 \log m + 24n^3 + 3n^2 + n}{m}.$$

Throughout, we will assume that m is lower bounded by the numerator, as otherwise, the bound is trivial.

For $i \neq j$, we again let \mathcal{E}^{ij} be the event that $A_i \not\prec_{\pi_i}^{\text{sd}} A_j$. Unlike in Round Robin, the give-away phase makes it possible that lower-indexed agents envy higher-indexed agents, so \mathcal{E}^{ij} is defined for all agents $i \neq j$. We again wish to upper bound $\Pr \left[\bigcup_{i \in \mathcal{N}} \bigcup_{j: j \neq i} \mathcal{E}^{ij} \right]$. We reuse the notation $g \succ_i g'$ to denote preferences under π_i , and b_{jk}^i for i 's k 'th favorite item in A_j . We decompose \mathcal{E}^{ij} into $\bigcup_{k=1}^q \mathcal{E}_k^{ij}$ as before, where \mathcal{E}_k^{ij} is the event that $b_{jk}^i \succ_i b_{ik}$. Using a union bound, we wish to upper bound

$$\sum_{i \in \mathcal{N}} \Pr \left[\bigcup_{k \leq q} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \right].$$

We let γ_{ij} for $i \neq j$ be the item given from i to j in the give-away phase. We use \tilde{P}_t for $0 \leq t \leq n(n-1)$ to denote the pool of available items after t giveaway steps. We then let P_t be the pool after t Round Robin steps (so $\tilde{P}_0 = \mathcal{M}$, $\tilde{P}_{n(n-1)} = P_0$, and $P_{m-n(n-1)} = \emptyset$). We let g_t be the item taken at the t 'th step of Round Robin. Therefore, $A_j = \{\gamma_{ij} \mid i \neq j\} \cup \{g_t \mid t \equiv j \pmod n\}$.

We fix an agent i and condition on π_i . The random process of selecting items is now similar to in Round Robin. For times t when γ_{ij} is given, γ_{ij} is the worst remaining item from \tilde{P}_{t-1} , and for $\gamma_{i'j}$ with $i' \neq i$ it is a uniformly selected item from \tilde{P}_{t-1} (from agent i 's perspective). During the Round Robin phase, for g_t with $t \equiv i \pmod n$, g_t is the best available item from P_{t-1} , and when $t \not\equiv i \pmod n$, g_t is a uniformly selected item from P_{t-1} . For the Round Robin phase, we again consider the sets R_ℓ of items picked in the Round Robin phase between i 's ℓ 'th and $(\ell+1)$ 'th pick, so $R_0 = \{g_1, \dots, g_{i-1}\}$, for $1 \leq \ell \leq q-n$, $R_\ell = \{g_{n(\ell-1)+i+1}, \dots, g_{n\ell+i-1}\}$, and $R_q = \{g_{n(q-1)+i+1}, \dots, g_m\}$. As with Round Robin, each item in R_ℓ is picked sequentially and uniformly at random from the remaining items. This allows us to analyze an equivalent two-step process wherein R_ℓ is first selected uniformly at random from P_t , after which each item in R_ℓ is matched to a (non- i) agent that is picked in the corresponding timesteps uniformly at random.

We now break down $\bigcup_{k \leq q}$ via a union bound into different groupings of k .

$$\sum_{i \in \mathcal{N}} \Pr \left[\bigcup_{k \leq q} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \right] = \sum_{i \in \mathcal{N}} \left(\Pr \left[\bigcup_{k: k < 32 \log m + 4n} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \right] + \Pr \left[\bigcup_{k: 32 \log m + 4n \leq k \leq q - 32 \log m - 5n} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \right] \right. \\ \left. + \Pr \left[\bigcup_{k: q - 32 \log m - 5n < k \leq q - 1} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \right] + \Pr \left[\bigcup_{k: k = q} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \right] \right).$$

Fix an agent i . The following lemmas handle each of the summands individually.

Lemma 6. $\Pr \left[\bigcup_{k: k < 32 \log m + 4n} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \right] \leq \frac{2048n^2 \log^2 m + 640n^3 \log m + 50n^4}{m}.$

Lemma 7. $\Pr \left[\bigcup_{k: 32 \log m + 4n \leq k \leq q - 32 \log m - 5n} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \right] \leq \frac{1}{m}.$

Lemma 8. $\Pr \left[\bigcup_{k: q - 32 \log m - 5n < k \leq q - 1} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \right] \leq \frac{65536n^2 \log^2 m + 24576n^3 \log m + 2304n^4 + 128n \log m + 24n^2 + n}{m}.$

Lemma 9. $\Pr \left[\bigcup_{k: k = q} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \right] \leq \frac{2n}{m}.$

Together, these imply that

$$\sum_{i \in \mathcal{N}} \Pr \left[\bigcup_{k \leq q} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \right] \\ \leq n \left(\frac{67584n^2 \log^2 m + 25216n^3 \log m + 2354n^4 + 128n \log m + 24n^2 + 3n + 1}{m} \right) \\ \leq \frac{67584n^3 \log^2 m + 25216n^4 \log m + 2354n^5 + 128n^2 \log m + 24n^3 + 3n^2 + n}{m} \in \tilde{O}_n \left(\frac{1}{m} \right). \quad \square$$

We prove Lemmas 6 to 9 in Appendix B.2.

4 Arbitrary Valuations

In this section, we consider the case of two agents having arbitrary set functions as their valuation functions. Such valuation functions need not be monotone and are general enough to capture all well-studied settings, including the fair division of goods, chores, or mixed manna under additive, subadditive, or superadditive preferences, etc. Our main result is that EF allocations exist with probability at least $1 - \frac{1}{m/2+1}$ when the number of items, m , is even. When the number of items is odd, valuation functions exist for which EF allocations cannot exist. For example, consider the case of two identical, additive agents whose value for every single item is equal to 1 (where random renaming does not change anything). For such a valuation function, if m is odd, then envy-free allocations do not exist. And, if m is even, every envy-free allocation is, in fact, *balanced*, i.e., each agent receives a bundle of the same size $m/2$. In Theorem 5, we show that balanced EF allocations exist with high probability for arbitrary valuation functions when m is even. As we'll see later in this section, looking for a balanced EF allocation will allow us to reduce our problem to a question about the size of automorphisms in k -uniform hypergraphs. First, in the following observation, we show that, without loss of generality, we may assume that each agent has a strict preference between a set and its complement.

Observation 1. *If any agent has a valuation function $v : 2^{\mathcal{M}} \rightarrow \mathbb{R}_{\geq 0}$ such that, for some $S \subseteq \mathcal{M}$, it holds that $v(S) = v(\bar{S})$, then an EF allocation exists with probability 1.*

Proof. Without loss of generality, let v be the valuation function of agent 1. For any permutation π_1 , agent 1 is indifferent between the bundles $\pi_1(S)$ and $\pi_1(\bar{S})$, therefore, either the allocation $(\pi_1(S), \pi_1(\bar{S}))$ or $(\pi_1(\bar{S}), \pi_1(S))$ is envy-free. \square

We henceforth assume that (i) the number of items, m , is even; and (ii) the valuation function v_i of each agent i is such that, for every subset S with $|S| = m/2$, $v_i(S) \neq v_i(\bar{S})$. Our proof is based on two key insights. First, (strict) preferences for bundles of size $m/2$ can be conveniently represented as uniform hypergraphs. Second, we can formulate questions about the probability of an envy-free allocation existing as questions about the number of hypergraph automorphisms.

4.1 Representing preferences as hypergraphs

We use the following hypergraph representation of arbitrary set function preferences: Given a valuation function v such that $v(S) \neq v(\bar{S})$ for all $S \subseteq \mathcal{M}, |S| = m/2$, define the hypergraph $H^v = (\mathcal{M}, E_v)$ where the set of edges $E_v := \{S \subseteq \mathcal{M} : |S| = m/2 \ \& \ v(S) > v(\bar{S})\}$. H^v is a $m/2$ -uniform hypergraph¹¹ with m vertices, one for each item, and $\binom{m}{m/2}/2$ edges, one for each set of size $m/2$ that is preferred to its complement.

Given a hypergraph $H = (\mathcal{M}, E)$ and a permutation $\pi : \mathcal{M} \rightarrow \mathcal{M}$, we will use $\pi(H) = (\mathcal{M}, E_\pi)$ to denote the hypergraph obtained by permuting the set of vertices by π , i.e., $E_\pi = \{\pi(S) : S \in E\}$. As an example, consider the valuations and corresponding hypergraphs shown in Figure 1. Each preferred bundle of size 2 corresponds to an edge in the hypergraph, representing the corresponding agent's preferences. Since $H^1 \neq H^2$ there exists an envy-free allocation, in this case $(\{1, 3\}, \{2, 4\})$.

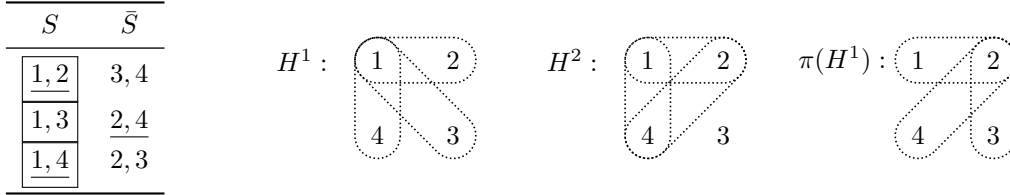


Figure 1: Example with $m = 4$ items and $n = 2$ agents where the hypergraphs corresponding to the valuations of agent 1, 2 are H^1, H^2 respectively. The leftmost table shows all bundles of size $m/2 = 2$ and their complements; the preferred bundles of agent 1 (2) are boxed (underlined, respectively). The rightmost figure shows the hypergraph $\pi(H^1)$ obtained by renaming H^1 using permutation $\pi = (2, 1, 3, 4)$.

The next lemma formalizes the relationship between envy-free allocations and the hypergraph representation of valuations.

Lemma 10. *Let $v_1, v_2 : 2^{\mathcal{M}} \rightarrow \mathbb{R}_{\geq 0}$ be two valuation functions with corresponding hypergraphs $H^{v_1} = (\mathcal{M}, E_{v_1})$ and $H^{v_2} = (\mathcal{M}, E_{v_2})$, respectively. If the hypergraphs are not identical, i.e. $H^{v_1} \neq H^{v_2}$, then an envy-free allocation exists.*

Proof. Since $H^{v_1} \neq H^{v_2}$, there must exist an edge $S \subseteq \mathcal{M}, |S| = m/2$, present in one of the hypergraphs that is not present in both. If $S \in E_{v_1}$ and $S \notin E_{v_2}$, then the allocation $\mathcal{A} = (S, \bar{S})$ is envy-free, since $v_1(S) > v_1(\bar{S})$ and $v_2(S) < v_2(\bar{S})$. Similarly, if $S \notin E_{v_1}$ and $S \in E_{v_2}$, then the allocation $\mathcal{B} = (\bar{S}, S)$ is envy-free. Thus, envy-free allocations exist when $H^{v_1} \neq H^{v_2}$. \square

4.2 Relating envy-freeness to hypergraph isomorphism

When agent i with valuation function v_i and corresponding hypergraph H^{v_i} renames the items by applying permutation π , then the hypergraph $H^{v_i^\pi}$ corresponding to the new valuation function v_i^π is $H^{v_i^\pi} = \pi(H^{v_i})$. That is, permuting the names of the items amounts to permuting the vertices of the hypergraph. This follows

¹¹A hypergraph $H = (V, E)$ is k -uniform when $|e| = k$ for all $e \in E$.

directly from the following sequence of equivalences: subset S is present as an edge in $H^{v_i^\pi}$ iff $v_i^\pi(S) > v_i^\pi(\bar{S})$ iff $v_i(\pi^{-1}(S)) > v_i(\pi^{-1}(\bar{S}))$ (from the definition of v_i^π) iff $\pi^{-1}(S)$ is present as an edge in H^{v_i} iff S is present as an edge in $\pi(H^{v_i})$.

In our setting, agents 1 and 2, with valuation functions v_1 and v_2 , apply permutations π_1 and π_2 to rename the items. As per Lemma 10, an envy-free allocation will exist whenever $\pi_1(H^{v_1}) \neq \pi_2(H^{v_2})$. Since applying permutations π_1, π_2 does not change the structure of the hypergraph, $\pi_1(H^{v_1}) = \pi_2(H^{v_2})$ is only possible when H^{v_1} and H^{v_2} are isomorphic (and therefore, envy-free allocations trivially exist with probability 1 whenever H^{v_1} and H^{v_2} are non-isomorphic).

Accordingly, we assume that H^{v_1} and H^{v_2} are isomorphic. Without loss of generality, we also assume that π_1 is the identity permutation. To see why, observe that for uniformly random π_1, π_2 , the composition $\pi_1^{-1}\pi_2$ is also uniformly random. We view agent 2's renaming of the items as the permutation group¹² acting on the set of all hypergraphs on m vertices, denoted $\mathcal{H}_m = \{H = (V, E) : |V| = m\}$ and recall relevant group theoretic concepts. The *orbit* of a hypergraph H is an equivalence class defined as $\mathcal{O}(H) = \{G \in \mathcal{H}_m : \exists \text{ permutation } \pi, G = \pi(H)\}$. The permutation group partitions \mathcal{H}_m into ℓ orbits $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_\ell$, where any two hypergraphs are isomorphic if and only if they are in the same orbit. In Figure 1, H^1 and H^2 are not in the same orbit. Despite H^1 and $\pi(H^1)$ being isomorphic and in the same orbit, these valuations still permit an envy-free allocation.

Let \mathcal{O} be the orbit containing both H^{v_1} and H^{v_2} . Starting from $H \in \mathcal{O}$ and applying a random permutation π , the distribution of the permuted hypergraph $\pi(H)$ is also uniform over \mathcal{O} . To upper bound the probability of the non-existence of EF allocations, it is sufficient to lower bound all orbit sizes. Specifically, $\Pr_{\pi_2}[H^{v_1} = \pi_2(H^{v_2})] = 1/|\mathcal{O}|$. The orbit-stabilizer theorem [DF04] relates a graph's orbit size to the number of its stabilizers.

Theorem 3 (Orbit-stabilizer theorem [DF04]). *If a finite group Π acts on a set X , then for every $x \in X$, $|\Pi| = |\text{Orb}(x)| \cdot |\text{Stab}(x)|$, where $\text{Stab}(x) = \{\pi \in \Pi : \pi \cdot x = x\}$.*

In our context Π is the permutation group and $X = \mathcal{H}_m$. A permutation π is an automorphism (and also a stabilizer) of hypergraph H iff $\pi(H) = H$. Let $\text{aut}(H) := \{\text{permutation } \pi : \pi(H) = H\}$ be the set of all automorphisms. Now $m! = |\mathcal{O}(H)| \cdot |\text{aut}(H)|$ by Theorem 3. We now upper bound the number of automorphisms of a hypergraph H .

Theorem 4. *For any non-empty and non-complete k -uniform hypergraph H on m vertices, the number of automorphisms of H satisfies $|\text{aut}(H)| \leq \frac{m!}{m-k+1}$.*

Proof. We prove this by induction on k . Our induction hypothesis is as follows: for a given k , the number of automorphisms of a non-empty and non-complete k -uniform hypergraph on m vertices is at most $\frac{m!}{m-k+1}$.

Base Case: $k = 1$. Fix an arbitrary m . A 1-uniform hypergraph $H = (\mathcal{M}, E)$ is simply a selection of some ℓ singleton sets of vertices: for all $S \in E$, we have $|S| = 1$. Since H is non-empty and non-complete, we have $0 < \ell < m$. Note that, any automorphism π of H must map any vertex that forms (resp. does not form) an edge to another vertex that forms (resp. does not form) an edge. Since there are ℓ vertices that form an edge and $m - \ell$ vertices that don't form an edge, we have

$$|\text{aut}(H)| \leq \ell!(m - \ell)! = \frac{m!}{\binom{m}{\ell}} \leq \frac{m!}{m} = \frac{m!}{m - k + 1}.$$

Induction Step: Now, assume that the induction hypothesis is true for k . Fix an arbitrary m and let $H = (\mathcal{M}, E)$ be a non-empty, non-complete $(k + 1)$ -uniform hypergraph on m vertices. For any vertex $i \in \mathcal{M}$, define H_i to be the k -uniform hypergraph induced by the vertex i . That is, $H_i = (V_i, E_i)$ where $V_i = \mathcal{M} \setminus \{i\}$ and $E_i = \{e \setminus \{i\} : e \in E, i \in e\}$. We consider two cases, based on whether the hypergraphs $\{H_i\}_{i \in \mathcal{M}}$ are all isomorphic to each other or not.

First, suppose the hypergraphs $\{H_i\}_{i \in \mathcal{M}}$ are not isomorphic to each other. Define $I_1 = \{j \in \mathcal{M} : \exists \text{ permutation } \pi \text{ satisfying } \pi(H_j) = H_1\}$ as the set of vertices whose corresponding hypergraph is isomorphic

¹²Recall that the elements of the permutation group are all permutations $\pi : \mathcal{M} \rightarrow \mathcal{M}$ and the group operation which combines two group elements is the function composition.

to H_1 and let $\ell = |I_1|$. $1 \in I_1$, therefore $\ell > 0$. Since not all H_i 's are isomorphic to each other we have $I_1 \subsetneq \mathcal{M}$, and thus $\ell < m$. In any automorphism π , every vertex in I_1 must be mapped to a vertex in I_1 . Hence, by the same argument as in the base case, the total number of automorphisms of H is at most

$$|\text{aut}(H)| \leq \ell!(m - \ell)! \leq \frac{m!}{m} \leq \frac{m!}{m - k + 1}.$$

Next, suppose the hypergraphs $\{H_i\}_{i \in \mathcal{M}}$ are isomorphic to each other. Because H is non-empty and non-complete and $k \geq 2$, each H_i cannot be empty or complete. That is, the hypergraphs $\{H_i\}_{i \in \mathcal{M}}$ are non-empty, non-complete, k -uniform, and have $m - 1$ vertices. Therefore, as per the induction hypothesis, we have $|\text{aut}(H_j)| \leq \frac{(m-1)!}{(m-1)-k+1}$ for all $j \in \mathcal{M}$. Finally, to count the total number of automorphisms of H , notice that there are exactly m ways to map vertex 1 to another vertex $j \in \mathcal{M}$. However, once this mapping is fixed, the hypergraph induced by the remaining vertices $\mathcal{M} \setminus \{j\}$, must get mapped to the hypergraph H_1 . Hence, the total number of automorphisms of H is at most

$$|\text{aut}(H)| \leq m \cdot \frac{(m-1)!}{(m-1)-k+1} = \frac{m!}{m - (k+1) + 1},$$

giving us the desired inequality. This completes the induction step. \square

We combine the previous observations and Theorem 4 to obtain the main result of this section.

Theorem 5. *For $n = 2$ agents, having arbitrary valuation functions, envy-free allocations exist with a probability of at least $1 - \frac{1}{m/2+1}$ when m is even. This probability is tight up to constants: for $n = 2$ agents, there exist additive valuations such that an envy-free allocation does not exist with probability $1/m$.*

Proof. From Observation 1 and Lemma 10, we only need to consider instances where agents have valuation functions v_1 and v_2 such that (i) H^{v_1} is isomorphic to H^{v_2} , and (ii) for all bundles $S \subseteq \mathcal{M}$ with $|S| = m/2$ we have $v(S) \neq v(\overline{S})$; if either (i) or (ii) is not satisfied, EF allocations will exist with probability 1. We can therefore represent the valuation functions by the corresponding hypergraphs H^{v_1} and H^{v_2} . Since H^{v_1} is isomorphic to H^{v_2} , instead of considering renaming H^{v_1} and H^{v_2} by random permutations $\hat{\pi}_1$ and $\hat{\pi}_2$ respectively, we can simply consider v_1 as fixed and permuting H^{v_2} by $\pi = \hat{\pi}_1^{-1}\hat{\pi}_2$.

By Lemma 10, envy-free allocations do not exist with probability at most $\Pr_\pi[H^{v_1} = \pi(H^{v_2})]$. If both hypergraphs belong to the orbit \mathcal{O} , i.e., $H^{v_1}, H^{v_2} \in \mathcal{O}$, then $\Pr_\pi[H^{v_1} = \pi(H^{v_2})] = \frac{1}{|\mathcal{O}|} = \frac{|\text{aut}(H^{v_1})|}{m!}$; the last inequality follows from the orbit-stabilizer theorem [DF04]. As the final step, we can use Theorem 4 with $k = m/2$ to obtain the bound

$$\Pr_\pi[H^{v_1} = \pi(H^{v_2})] = \frac{|\text{aut}(H^{v_1})|}{m!} \leq \frac{1}{m - (m/2) + 1} = \frac{1}{m/2 + 1}.$$

Therefore, envy-free allocations exist with a probability of at least $1 - \frac{1}{m/2+1}$.

To conclude the proof of the theorem, we show an upper bound on the probability of existence. Consider the case of two identical valuation functions that assign a positive value only to a single item (the same item for both agents). Any set that contains this item is preferred to its complement. For such an instance, with probability $1/m$ (over item renaming), this single item liked by both of the agents will be the same, and envy-free allocations won't exist. \square

5 Conclusion

In this paper, we study the existence of envy-free allocations beyond the case of additive valuations. We introduce a simple model to study this question: starting from a worst-case valuation function, randomly rename the items. We show that, in this model, if valuations are order-consistent (a valuation class general enough to include additive, unit-demand, budget additive, single-minded, etc), even sd-envy-free allocations

exist with high probability. In fact, a simple Round-Robin process will output such an allocation with high probability. Our bound on the probability is asymptotically tight, but we can improve upon it for the important case of a constant number of agents, using a variation of Round-Robin, that might be of independent interest. For arbitrary valuations, we show a positive result for the case of $n = 2$ agents. Our proof reduces the question about the existence of envy-free allocations to a question about the number of automorphisms in a uniform hypergraph.

An important problem we leave open is whether envy-free allocations exist for general valuation functions for $n > 2$ agents. Our approach of viewing valuations as hypergraphs immediately fails since now an allocation is not just a set and its complement. Interestingly, the existence of envy-free allocations in our model does not seem to become any easier, even if one is willing to make fairly common structural assumptions on the valuation functions. For example, one might attempt to prove Theorem 5 for, e.g., monotone submodular functions (where strong concentration results are readily available [Von10]), hoping to get a proof for $n = 2$ that can extend to more agents. In Section 4 we show that corresponding to a valuation function, we can construct a unique hypergraph; we complement this in Appendix A by showing that for any H we can construct a monotone submodular function v such that $H^v = H$. This implies that, in a sense, the general existence of balanced EF allocations for two agents reduces to the existence of EF allocations for monotone submodular valuation functions.

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A Preference Hypergraphs and Submodular valuations

We conclude by showing that the existence of balanced envy-free allocations for arbitrary valuation functions can, in a sense, be reduced to the class of submodular valuation functions.

Proposition 1. *For any valuation function v and corresponding hypergraph $H^v = (\mathcal{M}, E)$ such that, for all $S \subseteq \mathcal{M}$ of size $m/2$ exactly one of S or \bar{S} is in E , there exists a monotone, submodular valuation function v' so that $H^v = H^{v'}$.*

Proof. Construct v' as follows:

$$v'(S) = \begin{cases} |S| & \text{if } |S| < m/2 \\ m/2 - 1/2 & \text{if } |S| = m/2 \text{ and } \bar{S} \in E \\ m/2 - 1/4 & \text{if } |S| = m/2 \text{ and } S \in E \\ m/2 & \text{if } |S| > m/2. \end{cases}$$

It is straightforward to verify that v' is submodular and that $H^{v'} = H^v$. □

B Omitted proofs

Proof of Claim 1. Using Gautchi's inequality we have that

$$\begin{aligned}
\frac{2}{n} \cdot \frac{\Gamma(k+1/n)\Gamma(q-k+1/n)}{\Gamma(q+1/n)} &= \frac{2}{n} \cdot \frac{\Gamma(k+1)\Gamma(q-k+1)}{\Gamma(q+1)} \cdot \frac{\Gamma(k+1/n)}{\Gamma(k+1)} \cdot \frac{\Gamma(q-k+1/n)}{\Gamma(q-k+1)} \cdot \frac{\Gamma(q+1)}{\Gamma(q+1/n)} \\
&\leq \frac{2}{n} \cdot \frac{k!(q-k)!}{q!} \frac{1}{k^{1-1/n}} \frac{1}{(q-k)^{1-1/n}} (q+1)^{1-1/n} \\
&= \frac{2}{n} \frac{1}{\binom{q}{k}} \left(\frac{q+1}{k(q-k)} \right)^{1-1/n} \\
&< \frac{2}{n} \frac{1}{\binom{q}{k}} \left(\frac{2q}{q/2} \right)^{1-1/n} \\
&\leq \frac{2}{n} \frac{1}{\binom{q}{k}} 4^{1-1/n} \\
&\leq \frac{8}{n} \cdot \frac{1}{\binom{q}{k}}
\end{aligned}$$

where the third inequality holds because $q+1 \leq 2q$, both $k \geq 1$, $(q-k) \geq 1$, and at least one k and $(q-k)$ is $\geq q/2$. \square

Proof of Claim 3. Recall that $b_{iq} = g_{n(q-1)+i}$. Further, $g_{n(q-1)+i} \in L$ requires that $P_{n(q-1)+i-1} \subseteq L$, otherwise an item $g \notin L$, which i prefers to any item in L , is available (and i would have preferred to choose it). Since $|P_{n(q-1)+i-1}| = m - (n(q-1) + i - 1) = n - i + 1$, we have

$$\Pr[b_{iq} \in L] = \Pr[P_{n(q-1)+i-1} \subseteq L] = \sum_{S \in \binom{L}{n+1-i}} \Pr[P_{n(q-1)-i+1} = S].$$

Fixing an arbitrary $S \in \binom{L}{n+1-i}$ we have

$$\begin{aligned}
\Pr[P_{n(q-1)-i+1} = S] &= \Pr \left[\bigcap_{t=1}^{n(q-1)+i-1} \{g_t \notin S\} \right] \\
&= \prod_{t=1}^{n(q-1)+i-1} \Pr \left[g_t \notin S \mid \bigcap_{t'=1}^{t-1} g_{t'} \notin S \right] \\
&= \prod_{t=1}^{n(q-1)+i-1} \Pr[g_t \notin S \mid S \subseteq P_{t-1}].
\end{aligned}$$

For timesteps t where i made a pick, i.e., $t \equiv i \pmod{n}$, $\Pr[g_t \notin S \mid S \subseteq P_{t-1}]$ is hard to compute, as it depends on exactly which items are available. In these cases, we use the trivial upper bound of 1. For other timesteps t , g_t is a uniformly random item from P_{t-1} , so $\Pr[g_t \notin S \mid S \subseteq P_{t-1}] = 1 - \frac{|S|}{|P_{t-1}|} = 1 - \frac{n+1-i}{m+1-t} = \frac{m+1-t-(n+1-i)}{m+1-t}$. It follows that

$$\Pr[P_{n(q-1)-i+1} = S] \leq \prod_{t=1: t \not\equiv i \pmod{n}}^{n(q-1)+i-1} \frac{m+1-t-(n+1-i)}{m+1-t}.$$

To ease notation, we perform a change of variables to $t' = n(q-1) + i - t$, i.e., we reverse the product. The condition $t \not\equiv i \pmod{n}$ becomes $t' \not\equiv 0 \pmod{n}$, and we have $t' = n(q-1) + i - t = m+1-t-(n+1-i)$.

Therefore, we can simplify to

$$\begin{aligned} \prod_{t'=1:t' \not\equiv 0 \pmod n}^{n(q-1)+i-1} \frac{t'}{t' + (n+1-i)} &= \frac{\prod_{t'=1}^{n(q-1)+i-1} \frac{t'}{t' + (n+1-i)}}{\prod_{t'=1:t' \equiv 0 \pmod n}^{n(q-1)+i-1} \frac{t'}{t' + (n+1-i)}} \\ &= \prod_{t'=1}^{n(q-1)+i-1} \frac{t'}{t' + (n+1-i)} \cdot \prod_{t'=1:t' \equiv 0 \pmod n}^{n(q-1)+i-1} \frac{t' + (n+1-i)}{t'}. \end{aligned}$$

Observing that $n(q-1) + i - 1 + (n+1-i) = qn = m$, the first product is exactly equal to

$$\frac{(m - (n+1-i))!(n+1-i)!}{m!} = \frac{1}{\binom{m}{n+1-i}} \leq \frac{1}{(m - (n+1-i))^{n+1-i}} \leq \frac{1}{(m-n)^{n+1-i}} \leq_{(m \geq 2n)} \left(\frac{2}{m}\right)^{n+1-i}.$$

The second product simplifies, as we only need to consider values $t' = \ell n$ for $1 \leq \ell \leq q-1$. Hence,

$$\prod_{t'=1:t' \equiv 0 \pmod n}^{n(q-1)+i-1} \frac{t' + (n+1-i)}{t'} = \prod_{\ell=1}^{q-1} \frac{\ell n + (n+1-i)}{\ell n} = \prod_{\ell=1}^{q-1} \frac{\ell + \frac{n+1-i}{n}}{\ell} = \frac{(1 + \frac{n+1-i}{n})^{q-1}}{(q-1)!},$$

where the notation $(x)^{\bar{r}}$ represents the rising factorial $(x)^{\bar{r}} = x(x+1)\cdots(x+r-1)$. We use the fact that $(x)^{\bar{r}} = \frac{\Gamma(x+r)}{\Gamma(x)}$ to get that the above is equal to,

$$\frac{\Gamma(q + \frac{n+1-i}{n})}{\Gamma(1 + \frac{n+1-i}{n})\Gamma(q)}.$$

Similarly to Claim 1, using the fact that the Gamma function is lower bounded by 1/2 and Gautchi's inequality [Gau59], along with the assumption that $m \geq 2n$, the above is at most

$$\frac{2\Gamma(q + \frac{n+1-i}{n})}{\Gamma(q)} \leq 2 \left(q + \frac{n+1-i}{n}\right)^{\frac{n+1-i}{n}} \leq 2(q+1)^{\frac{n+1-i}{n}} \leq 2m^{\frac{n+1-i}{n}} = 2(m^{1/n})^{n+1-i}.$$

Combining with the previous bounds, we obtain for any $S \in \binom{[L]}{n+1-i}$ that, $\Pr [P_{n(q-1)-i+1} = S] \leq \left(\frac{2}{m}\right)^{n+1-i} \cdot 2(m^{1/n})^{n+1-i} = 2 \left(\frac{2}{m^{1-1/n}}\right)^{n+1-i}$. The number of possible choices of S is

$$\binom{|L|}{n+1-i} \leq |L|^{n+1-i} \leq (4n \log m)^{n+1-i}. \quad (8)$$

Union bounding over all of these choices, we get that

$$\Pr [b_{iq} \in L] = \Pr [P_{n(q-1)-i+1} \subseteq L] \leq 2 \left(\frac{8n \log m}{m^{1-1/n}}\right)^{n+1-i},$$

which concludes the proof of Claim 3. \square

B.1 Missing details from the proof of Lemma 2

Here, we flesh out the missing details from the proof of Lemma 2.

For $k < q$, the proof of Lemma 4 goes through almost identically, to give us $\Pr [\mathcal{E}_k^{ij}] \leq \frac{8}{\binom{q}{k}}$. The only difference is that the equality in $m - ((\ell-1)n + j - 1) = (q-\ell)n + n - j + 1$ of Equation (3) becomes a weak inequality; the inequality is in the right direction for the same conclusion to hold.

For $k = q$, the proof of Lemma 5, the analysis is very similar, even though more care is needed. To ensure that there are at least $3 \log m$ items remaining when lower bounding the expectation in Equation (6), since $|R_0| + |R_q|$ is only upper bounded by $(n-1) + r \leq 2n-1$, we slightly change L , which is now defined as the bottom $\lceil 3n \log m \rceil + 2n-1$ items. It follows that $|L| \leq 5n \log m$. A second change is in the proof of Claim 3. $|P_{n(q-1)+i-1}|$, the number of items remaining when i makes their q 'th pick, is now $n-i+1+r$. The analysis continues to go through by replacing occurrences of $n-i+1$ with $n-i+1+r$. In the change of variables, we have $t' = n(q-1) + i - t = m+1-t - (n+1-i+r)$. This means that Inequality (8) becomes

$$\binom{|L|}{n+1-i+r} \leq |L|^{n+1-i+r} \leq (5n \log m)^{n+1-i+r},$$

and Inequality (5) becomes

$$\Pr \left[\bigcup_{j:j < i} \mathcal{E}_q^{ij} \right] \leq \frac{n}{m} + 2 \left(\frac{10n \log m}{m^{1-1/n}} \right)^{n+1-i+r}. \quad (9)$$

For $n = 2$, we will need a bound that is even stronger than (9). When there are two agents the only envy event we need to consider is \mathcal{E}_q^{21} . We claim that we need only include i 's worst $1+r \leq 3$ items in L to get $\Pr[\mathcal{E}_q^{21} \mid b_{2q} \notin L] = 0$. Indeed, suppose $b_{2q} \notin L$, so b_{2q} is not one of the 3 worst items according to π_2 . Since $|R_q| \leq 2$ and $L \cap A_2^q = \emptyset$, it must be that $A_1^q \cap L \neq \emptyset$. We conclude $b_{1q} \in L$, so $b_{2q} \succ_2 b_{1q}$ and event \mathcal{E}_q^{21} does not occur. Therefore, when $n = 2$,

$$\sum_{i \in \mathcal{N}} \Pr \left[\bigcup_{j:j < i} \mathcal{E}_q^{ij} \right] = \Pr[\mathcal{E}_q^{21}] \leq 2 \left(\frac{3}{m^{1-1/n}} \right)^{n+1-i+r} \leq 2 \left(\frac{2n}{m^{1/2}} \right)^{1+r} \quad (10)$$

We are now ready to upper bound the probability that \mathcal{A}^q is not SD-EF.

$$\begin{aligned} \Pr \left[\bigcup_{i,j} \mathcal{E}^{ij} \right] &\leq \sum_{i \in \mathcal{N}} \sum_{k=1}^q \Pr \left[\bigcup_{j:j < i} \mathcal{E}_k^{ij} \right] \\ &\leq \sum_{i \in \mathcal{N}} \left(\sum_{k=1}^{q-1} \Pr \left[\bigcup_j \mathcal{E}_k^{ij} \right] + \Pr \left[\bigcup_j \mathcal{E}_q^{ij} \right] \right) \\ &\leq n \sum_{k=1}^{q-1} \Pr \left[\bigcup_j \mathcal{E}_k^{ij} \right] + \sum_{i \in \mathcal{N}} \Pr \left[\bigcup_j \mathcal{E}_q^{ij} \right] \\ &\leq 8n \sum_{k=1}^{q-1} \frac{1}{\binom{q}{k}} + \sum_{i \in \mathcal{N}} \Pr \left[\bigcup_j \mathcal{E}_q^{ij} \right]. \end{aligned}$$

Since $qn \geq m/2$ we can refine Inequality (2) to get $\sum_{k=1}^{q-1} \frac{1}{\binom{q}{k}} \leq \frac{4}{q} \leq \frac{8n}{m}$, so the first term is at most $\frac{64n^2}{m}$.

When $n = 2$, the second term is bounded by $2 \left(\frac{2n}{m^{1/2}} \right)^{1+r}$ by (10). By our assumption $m \geq 2000n^2$ and $r \geq 1$, so $\frac{2n}{m^{1/2}} \leq 1$ and

$$2 \left(\frac{2n}{m^{1/2}} \right)^{1+r} \leq 2 \left(\frac{2n}{m^{1/2}} \right)^{1+1} = \frac{8n^2}{m},$$

from which it follows that $\Pr \left[\bigcup_{i,j} \mathcal{E}^{ij} \right] \leq \frac{64n^2}{m} + \frac{8n^2}{m} = \frac{72n^2}{m} \leq \frac{2066n^2}{m}$.

Next, we show the same bound for $n \geq 3$. Here

$$\Pr \left[\bigcup_j \mathcal{E}_q^{ij} \right] \leq \frac{n^2}{m} + 2 \sum_{i \in \mathcal{N}} \left(\frac{10n \log m}{m^{1-1/n}} \right)^{n+1-i+r}$$

$$\begin{aligned}
&\leq \frac{n^2}{m} + 2 \sum_{i=1}^{\infty} \left(\frac{10n \log m}{m^{1-1/n}} \right)^{i+r} \\
&\leq \frac{n^2}{m} + 2 \sum_{i=1}^{\infty} \left(\frac{10n \log m}{m^{2/3}} \right)^{i+r}.
\end{aligned}$$

We claim that

$$\frac{\log m}{m^{1/6}} \leq \frac{6}{e}.$$

To see why, observe that the derivative of $\frac{\log m}{m^{1/6}}$ is $\frac{6-\log(m)}{6m(7/6)}$ the function is maximized over positive reals exactly at $m = e^6$, where it takes value $\frac{6}{e}$. Substituting into the above,

$$\Pr \left[\bigcup_j \mathcal{E}_q^{ij} \right] \leq \frac{n^2}{m} + 2 \sum_{i=1}^{\infty} \left(\frac{60n}{em^{1/2}} \right)^{i+r}.$$

Since $(60/e)^2 \leq 500$ and we are restricted to $m \geq 2000n^2$, $\frac{60n}{em^{1/2}} \leq \sqrt{\frac{500}{2000}} = 1/2$. Using the fact that $r \geq 1$, this infinite series is at most

$$\frac{n^2}{m} + 4 \left(\frac{60n}{em^{1/2}} \right)^2 \leq \frac{2001n^2}{m}.$$

Hence,

$$\Pr \left[\bigcup_j \mathcal{E}_q^{ij} \right] \leq \frac{2066n^2}{m},$$

as needed.

B.2 Missing lemmas from Section 3.4

Proof of Lemma 6. Let T be the top $\lfloor 32 \log m + 4n \rfloor$ items according to π_i . Notice that if $T \subseteq A_i$, $b_{ik} \succ b_{jk}$ for all $k \leq 32 \log m + 4n$. More formally, this means that,

$$\Pr \left[\bigcup_{k:k < 32 \log m + 4n} \bigcup_{j:j \neq i} \mathcal{E}_k^{ij} \right] \leq \Pr [T \not\subseteq A_i],$$

so we directly upper bound the latter. For $T \not\subseteq A_i$ to occur, it must be the case that some other agent j received an item in T before i made their $(\lfloor 32 \log m + 4n \rfloor)$ 'th pick. Since we are assuming $m \geq 32 \log m + 4n + n^2 \geq |T| + n^2$, i would never give away an item in T during the giveaway phase as a worse item must be available. Hence, the only way that another agent receives an item in T is if are given it by a non- i agent in the giveaway phase or they pick it in the Round Robin phase before time $n(|T| - 1) + i \leq n|T|$. From i 's perspective all of these items are chosen uniformly at random from the pool. There are at most $(n-1)(|T| + n) \leq n|T| + n^2 \leq 32n \log m + 5n^2$ such items. When they are taken, the pool is always of size at least $m - (32n \log m + 5n^2)$. Hence, each is selected with probability at most $\frac{32n \log m + 5n^2}{m - 32n \log m - 5n^2}$. Since we are assuming $m \geq 64n \log m + 10n^2$, this probability is at most $\frac{64n \log m + 10n^2}{m}$. Union bounding over the at most $32n \log m + 5n^2$ choices of k , this implies that the total probability is at most

$$\begin{aligned}
\Pr \left[\bigcup_{k:k < 32 \log m + 4n} \bigcup_{j:j \neq i} \mathcal{E}_k^{ij} \right] &\leq \frac{(64n \log m + 10n^2)(32n \log m + 5n^2)}{m} \\
&= \frac{2048n^2 \log^2 m + 640n^3 \log m + 50n^4}{m} \quad \square
\end{aligned}$$

Proof of Lemma 7. Here, we upper bound each $\Pr \left[\mathcal{E}_k^{ij} \right]$ individually and union bound over the at most q choices of k and n choices of j . Fix agent $j \neq i$ and k such that $32 \log m + 4n \leq k \leq q - 32 \log m - 5n$. We first consider $j < i$ and later show how to extend it to $j > i$. Notice that $b_{ik} \succeq_i g_{(k-1)n+i}$, i.e., i 's k 'th favorite item is at least as good as their k 'th pick in the Round Robin phase (though it may be better if i was lucky during the giveaway phase). Additionally, if $b_{jk} \succ_i b_{ik} \succeq_i g_{(k-1)n+i}$, this implies that A_j contains at least k items strictly preferred to $g_{(k-1)n+i}$. None of these could have been chosen after time $(k-1)n+i$ in the Round Robin phase as $g_{(k-1)n+i}$ was the best available item from $P_{(k-1)n+i-1}$. Additionally, $g_{(k-1)n+i} \succ \gamma_{ij}$ (the item i gave to j) as γ_{ij} was the worst available item at a time when $g_{(k-1)n+i}$ was available. Hence, for \mathcal{E}_k^{ij} to hold, at least k of $\{\gamma_{i'j} \mid i' \neq i, j\} \cup \{g_j, \dots, g_{(k-1)n+j}\}$ are preferred to $g_{(k-1)n+i}$. Since $|\{\gamma_{i'j} \mid i' \neq i, j\}| = n-2$, this implies that $|\{\ell \mid g_{(\ell-1)n+j} \succ_i g_{(k-1)n+i}\}| \geq k - (n-2)$. More formally, we have that

$$\Pr \left[\mathcal{E}_k^{ij} \right] \leq \Pr \left[\sum_{\ell=1}^k \mathbb{I}[g_{(\ell-1)n+j} \succ_i g_{(k-1)n+i}] \geq k - (n-2) \right]$$

As before, a necessary condition for $g_{(\ell-1)n+j} \succ_i g_{(k-1)n+i}$ is $g_{(\ell-1)n+j} \succeq_i P_{(\ell-1)n+j-1}^{((k-\ell)n+i-j)}$, hence,

$$\begin{aligned} \Pr \left[\sum_{\ell=1}^k \mathbb{I}[g_{(\ell-1)n+j} \succ_i g_{(k-1)n+i}] \geq k - (n-2) \right] &\leq \Pr \left[\sum_{\ell=1}^k \mathbb{I}[g_{(\ell-1)n+j} \succeq_i P_{(\ell-1)n+j-1}^{((k-\ell)n+i-j)}] \geq k - (n-2) \right] \\ &= \Pr \left[\sum_{\ell=1}^k \mathbb{I}[g_{(\ell-1)n+j} \not\prec_i P_{(\ell-1)n+j-1}^{((k-\ell)n+i-j)}] \leq n-2 \right]. \end{aligned}$$

Now each of these indicators is an independent Bernoulli random variable. This means we can use a Chernoff bound on the sum. To do so, we first bound the expectation of the sum. We have that

$$\begin{aligned} \mathbb{E} \left[\sum_{\ell=1}^k \mathbb{I}[g_{(\ell-1)n+j} \not\prec_i P_{(\ell-1)n+j-1}^{((k-\ell)n+i-j)}] \right] &= \sum_{\ell=1}^k \Pr \left[g_{(\ell-1)n+j} \not\prec_i P_{(\ell-1)n+j-1}^{((k-\ell)n+i-j)} \right] \\ &= \sum_{\ell=1}^k 1 - \frac{(k-\ell)n+i-j}{|P_{(\ell-1)n+j-1}|} \\ &= \sum_{\ell=1}^k 1 - \frac{(k-\ell)n+i-j}{n(q-(n-1)) - ((\ell-1)n+j-1)} \\ &= \sum_{\ell=1}^k 1 - \frac{(k-\ell)n+i-j}{n(q-n-\ell)+1-j} \\ &= \sum_{\ell=1}^k \frac{n(q-n-k)+1-i}{n(q-n-\ell)+1-j} \\ &\geq \sum_{\ell=1}^k \frac{n(q-n-k)+1-i+(i-1)}{n(q-n-\ell)+1-j+(i-1)} \\ &= \sum_{\ell=1}^k \frac{n(q-n-k)}{n(q-n-\ell)+i-j} \\ &\geq \sum_{\ell=1}^k \frac{n(q-n-k)}{n(q-n-\ell)+n} \\ &\geq \sum_{\ell=1}^k \frac{q-n-k}{q-n-\ell+1} \end{aligned}$$

$$\begin{aligned}
&\geq k \cdot \frac{q-n-k}{q-n+1} \\
&\geq \frac{(q-n-k)k}{q-n+1} + \frac{(q-n-k)}{q-n+1} - 1 \\
&= \frac{(q-n-k)(k+1)}{q-n+1} - 1 \\
&\geq \frac{\max(q-n-k, k+1) \cdot \min(q-n-k, k+1)}{q-n+1} - 1 \\
&\geq \frac{1}{2} \cdot \min(q-n-k, k+1) - 1 \\
&\geq 16 \log m + 2n - 1.
\end{aligned}$$

Let $\mu = \mathbb{E} \left[\sum_{\ell=1}^k \mathbb{I}[g_{(\ell-1)n+j} \not\prec_i P_{(\ell-1)n+j-1}^{((k-\ell)n+i-j)}] \right]$. The above implies that $(n-2) \leq \frac{1}{2}\mu$. Hence, a Chernoff bound implies that

$$\begin{aligned}
\Pr \left[\sum_{\ell=1}^k \mathbb{I}[g_{(\ell-1)n+j} \not\prec_i P_{(\ell-1)n+j-1}^{((k-\ell)n+i-j)}] \leq n-2 \right] &\leq \Pr \left[\sum_{\ell=1}^k \mathbb{I}[g_{(\ell-1)n+j} \not\prec_i P_{(\ell-1)n+j-1}^{((k-\ell)n+i-j)}] \leq \frac{1}{2}\mu \right] \\
&\leq \exp \left(-\frac{\left(\frac{1}{2}\right)^2 \mu}{2} \right) \\
&= \exp \left(-\frac{\mu}{8} \right) \\
&\leq \exp(-2 \log m) = \frac{1}{m^2}.
\end{aligned}$$

All together, this implies that for $j < i$, $\Pr \left[\mathcal{E}_k^{ij} \right] \leq 1/m^2$.

For $j > i$, the argument can be modified as follows. Notice that j 's k 'th pick will be after i 's k 'th pick, hence, we need only sum over ℓ from 1 to $k-1$. This gives

$$\begin{aligned}
\Pr \left[\sum_{\ell=1}^{k-1} \mathbb{I}[g_{(\ell-1)n+j} \succ_i g_{(k-1)n+i}] \geq k - (n-2) \right] &\leq \Pr \left[\sum_{\ell=1}^{k-1} \mathbb{I}[g_{(\ell-1)n+j} \succeq_i P_{(\ell-1)n+j-1}^{((k-\ell)n+i-j)}] \geq k - (n-2) \right] \\
&= \Pr \left[\sum_{\ell=1}^{k-1} \mathbb{I}[g_{(\ell-1)n+j} \not\prec_i P_{(\ell-1)n+j-1}^{((k-\ell)n+i-j)}] \leq n-3 \right].
\end{aligned}$$

The analysis of the expectation goes through (replacing k with $k-1$) since we never use that $j < i$, at least until getting the expectation is lower bounded by

$$\sum_{\ell=1}^{k-1} \frac{q-n-k}{q-n-\ell+1}.$$

Now, we can continue with

$$\sum_{\ell=1}^{k-1} \frac{q-n-k}{q-n-\ell+1} \geq \sum_{\ell=1}^k \frac{q-n-k}{q-n-\ell+1} - 1,$$

at which point the remainder of the analysis continues to hold (off by one) until $\mu \geq 16 \log m + 2n - 2$. This is still sufficient for $(n-3) \leq \frac{1}{2}\mu$ and $\mu \geq 16 \log m$, which is all that was needed for the Chernoff bound to hold. This means that $\Pr \left[\mathcal{E}_k^{ij} \right] \leq 1/m^2$ for $j > i$ as well.

Union bounding over the q possible choices of k and n possible choices of j , we get that

$$\Pr \left[\bigcup_{k: 32 \log m + 4n \leq k \leq q - 32 \log m - 5n} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \right] \leq \frac{qn}{m^2} = \frac{1}{m}. \quad \square$$

Proof of Lemma 8. Let L be the last $\lfloor 64n \log m + 12n^2 \rfloor$ items according to π_i . Just as in Lemma 5, we will decompose

$$\begin{aligned} \Pr \left[\bigcup_{k: q - 32 \log m - 5n < k \leq q - 1} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \right] &\leq \Pr \left[\bigcup_{k: q - 32 \log m - 5n < k \leq q - 1} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \mid b_{(q-1)i} \notin L \right] + \Pr [b_{(q-1)i} \in L] \\ &\leq \sum_{j: j \neq i} \Pr \left[\bigcup_{k: q - 32 \log m - 5n < k \leq q - 1} \mathcal{E}_k^{ij} \mid b_{(q-1)i} \notin L \right] + \Pr [b_{(q-1)i} \in L]. \end{aligned}$$

We upperbound each of these summands. We begin with $\Pr [b_{(q-1)i} \in L]$. We again slit decompose this depending on if i received any goods in L during the giveaway phase.

$$\Pr [b_{(q-1)i} \in L] \leq \Pr [b_{(q-1)i} \mid \{\gamma_{ji} \mid j \neq i\} \cap L = \emptyset] + \Pr [\{\gamma_{ji} \mid j \neq i\} \cap L \neq \emptyset].$$

For $\Pr [\{\gamma_{ji} \mid j \neq i\} \cap L \neq \emptyset]$, recall that γ_{ji} are just random items from the pool from when the pool is of size at least $m - n^2$, hence, the probability each is an element of L is at most $\frac{|L|}{m - n^2}$. Using the assumption that $m \geq 2n^2$, we have that this is at most $\frac{2|L|}{m} \leq \frac{128n \log m + 20n^2}{m}$.

For $\Pr [b_{(q-1)i} \in L \mid \{\gamma_{ji} \mid j \neq i\} \cap L = \emptyset]$, since we are conditioning on i not being given any items in L during the giveaway phase, the only way that i will receive such items is if they pick them during round robin. For $b_{(q-1)i} \in L$, i must receive at least 2 items in L . This means that even when i made their second to last pick at Round Robin $g_{n(q-n-1)+i}$ the pool was $P_{n(q-n-1)+i-1}$, the only available items were from L , i.e., $P_{n(q-n-1)+i-1} \subseteq L$. Notice that $|P_{n(q-n-1)+i-1}| = m - n(n-1) - (n(q-n-1) + i - 1) = 2n - i + 1$. We will union bound over all possible sets $S \in \binom{L}{2n-i+1}$ analyzing the probability that $\Pr [P_{n(q-n-1)+i-1} = S \mid \{\gamma_{ji} \mid j \neq i\} \cap L \neq \emptyset]$.

We can decompose

$$\begin{aligned} \Pr [P_{n(q-n-1)+i-1} = S \mid \{\gamma_{ji} \mid j \neq i\} \cap L \neq \emptyset] \\ = \Pr [S \subseteq P_0 \mid \{\gamma_{ji} \mid j \neq i\} \cap L \neq \emptyset] \cdot \prod_{t=1}^{P_{n(q-n-1)+i-1}} \Pr [g_t \notin S \mid S \subseteq P_{t-1} \wedge \{\gamma_{ji} \mid j \neq i\} \cap L \neq \emptyset]. \end{aligned}$$

For the first term and all terms where $t \equiv i \pmod n$, this probability may be hard to compute, but we can upperbound it by 1. For all other terms with $t \not\equiv i \pmod n$,

$$\Pr [g_t \notin S \mid S \subseteq P_{t-1} \wedge \{\gamma_{ji} \mid j \neq i\} \cap L \neq \emptyset] = 1 - \frac{|S|}{|P_{t-1}|}.$$

Hence, this entire product is upper bounded by

$$\prod_{t=1: t \not\equiv i \pmod n}^{n(q-n-1)+i-1} \frac{n(q - (n-1)) + 1 - t - (2n - i + 1)}{n(q - (n-1)) + 1 - t}.$$

Notice that this is now the same analysis as the proof of Claim 3 with m replaced with $q(n-1)$ and $n+1-i$ replaced with $2n+1-i$. Hence, the above product is upperbounded by

$$2 \left(\frac{2}{(n(q - (n-1)))^{1-1/n}} \right)^{2n+1-i}.$$

Using the fact that $m \geq 2n^2$ so $n(q - (n - 1)) \geq m/2$, this is at most

$$2 \left(\frac{4}{m^{1-1/n}} \right)^{2n+1-i}.$$

Union bounding over the at most $\binom{|L|}{2n+1-i} \leq |L|^{2n+1-i} \leq (64n \log m + 10n^2)^{2n+1-i}$ subsets S , we get that

$$\Pr [b_{(q-1)i} \mid \{\gamma_{ji} \mid j \neq i\} \cap L = \emptyset] \leq \left(\frac{256n \log m + 48n^2}{m^{1-1/n}} \right)^{2n+1-i} \leq \left(\frac{256n \log m + 48n^2}{m^{1/2}} \right)^{2n+1-i}.$$

Since we have restricted to $m \geq (256n \log m + 40n^2)^2$, $m^{1/2} \geq 256n \log m + 48n^2$, so the inside is at most 1. Hence, since $2n + 1 - i \geq n + 1 \geq 2$, this is at most

$$\left(\frac{64n \log m + 12n^2}{m^{1/2}} \right)^2 = \frac{65536n^2 \log^2 m + 24576n^3 \log m + 2304n^4}{m}.$$

Hence, we have that

$$\Pr [b_{(q-1)i} \in L] \leq \frac{65536n^2 \log^2 m + 24576n^3 \log m + 2304n^4 + 128n \log m + 24n^2}{m}.$$

Finally, fix some agent $j \neq i$. We consider $\Pr \left[\bigcup_{k: q-32 \log m - 5n < k \leq q-1} \mathcal{E}_k^{ij} \mid b_{(q-1)i} \notin L \right]$. Notice that if $b_{(q-1)i} \notin L$, as long as $|A_j \cap L| \geq 32 \log m + 5n$, then for all $k \leq 32 \log m + 5n$, $b_{ik} \succeq_i b_{i(q-1)} \succ_i b_{jk}$, so \mathcal{E}_k^{ij} does not hold. Hence, we wish to upperbound the probability that $|A_j \cap L| < 32 \log m + 5n$. We use the second equivalent process, and consider sampling all of $\gamma_{i'j'}$ items along with sets R_0, \dots, R_q and goods picked by agent i , g_t with $t \equiv i \pmod n$, but we have not yet sampled which non- i agent receives which items in R_0, \dots, R_q . Notice that conditioned on $b_{i(q-1)} \notin L$, $|A_i \cap L| \leq 1$. Since $n(n-1)$ items are in the giveaway phase and $|R_0| + |R_q| = n-1$, this implies that $\left| L \cap \left(\bigcup_{\ell=1}^{q-1} R_\ell \right) \right| \geq |L| - n^2 \geq \lfloor 64n \log m + 11n^2 \rfloor \geq 64n \log m + 10n^2$. Notice that $|A_j \cap L| \geq \sum_{\ell=1}^{q-1} \mathbb{I}[A_j \cap R_\ell \cap L \neq \emptyset]$. Further, this is a sum of independent Bernoulli variables with $\Pr [A_j \cap R_\ell \cap L \neq \emptyset] = \frac{|R_\ell \cap L|}{n-1}$. Hence, $\mathbb{E}[\sum_{\ell=1}^{q-1} \mathbb{I}[A_j \cap R_\ell \cap L \neq \emptyset]] \geq \sum_{\ell=1}^{q-1} \frac{|R_\ell \cap L|}{n-1} = \frac{|L \cap (\bigcup_{\ell=1}^{q-1} R_\ell)|}{n-1} \geq 64 \log m + 10n$. We can then use a Chernoff bound to show that

$$\begin{aligned} \Pr [|A_j \cap L| \leq 32 \log m + 5n] &\leq \Pr \left[\sum_{\ell=1}^{q-1} \mathbb{I}[A_j \cap R_\ell \cap L \neq \emptyset] \leq 32 \log m + 5n \right] \\ &\leq \Pr \left[\sum_{\ell=1}^{q-1} \mathbb{I}[A_j \cap R_\ell \cap L \neq \emptyset] \leq \frac{1}{2} \mathbb{E} \left[\sum_{\ell=1}^{q-1} \mathbb{I}[A_j \cap R_\ell \cap L \neq \emptyset] \right] \right] \\ &\leq \exp \left(- \frac{(1/2)^2 \mathbb{E} \left[\sum_{\ell=1}^{q-1} \mathbb{I}[A_j \cap R_\ell \cap L \neq \emptyset] \right]}{2} \right) \\ &\leq \exp \left(- \frac{32 \log m}{8} \right) \leq \frac{1}{m^4} \leq \frac{1}{m}. \end{aligned}$$

Hence, we have that

$$\sum_{j: j \neq i} \Pr \left[\bigcup_{k: q-32 \log m - 5n < k \leq q-1} \mathcal{E}_k^{ij} \mid b_{(q-1)i} \notin L \right] \leq \frac{n}{m}.$$

Putting this together, we have that

$$\Pr \left[\bigcup_{k: q-32 \log m - 5n < k \leq q-1} \bigcup_{j: j \neq i} \mathcal{E}_k^{ij} \right]$$

$$\leq \frac{65536n^2 \log^2 m + 24576n^3 \log m + 2304n^4 + 128n \log m + 24n^2 + n}{m}. \quad \square$$

Proof of Lemma 9. Let L be the last $n-1$ items according to π_i . A sufficient condition for $\bigcup_{k:k=q} \bigcup_{j:j \neq i} \mathcal{E}_k^{ij} = \bigcup_{j:j \neq i} \mathcal{E}_q^{ij}$ to hold is that $\{\gamma_{ij} \mid i \neq j\} = L$, i.e., i is able to give away their bottom $n-1$ goods. In this case, $b_{iq} \succ_i L$ while $b_{jq} \in L$ for all $j \neq i$. For this to hold, the only necessary condition is that no other agent has given away a good in L prior to i giving them away. There are at most n^2 items given away before i is able to give, and each of these is an element of L with probability at most $\frac{|L|}{m-n^2}$. Since we are assuming that $M \geq 2n^2$, this is at most $\frac{2n}{m}$. Hence,

$$\Pr \left[\bigcup_{k:k=q} \bigcup_{j:j \neq i} \mathcal{E}_k^{ij} \right] \leq \frac{2n}{m}. \quad \square$$

C Non-existence of SD-EFX allocations

Complementing Theorem 1, in this section we show that SD-EFX allocations may not exist, even for additive valuations. Additionally, this also contrasts with the fact that the existence of EFX remains an elusive open problem.

We first consider an instance $m = 4$ items and $n = 2$ agents having identical additive valuation function $v : 2^{[4]} \rightarrow \mathbb{R}_{\geq 0}$. The function v is such that $v(\{1\}) = 4$, $v(\{2\}) = 1 + \epsilon$, $v(\{3\}) = 1$, and $v(\{4\}) = 1 - \epsilon$. Indeed, the only EFX allocations in this instance are $\mathcal{A} = (\{1\}, \{2, 3, 4\})$ and $\hat{\mathcal{A}} = (\{2, 3, 4\}, \{1\})$. Note that an SD-EFX allocation must also be EFX, therefore, if an SD-EFX allocation exists for this instance it must be \mathcal{A} or $\hat{\mathcal{A}}$.

Furthermore, if an allocation is SD-EFX, then it must also be EFX for any other instance in which both agents have the same preference order over the items. Now consider a second instance wherein both agents instead have the additive function \hat{v} such that $\hat{v}(\{1\}) = 4$, $\hat{v}(\{2\}) = 3$, $\hat{v}(\{3\}) = 2$, and $\hat{v}(\{4\}) = 1$. Note that the preference order of items is the same in v and \hat{v} , therefore, if \mathcal{A} (or $\hat{\mathcal{A}}$) is SD-EFX then it must be EFX for the second instance. It is easy to see that neither \mathcal{A} nor $\hat{\mathcal{A}}$ is EFX for the second instance. This shows that SD-EFX allocations do not exist, even for additive valuations.